CHARACTERIZATIONS OF HANKEL MULTIPLIERS

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Abstract. We give characterizations of radial Fourier multipliers as acting on radial L^p functions, $1 < p < 2d/(d+1)$, in terms of Lebesgue space norms for Fourier localized pieces of the convolution kernel. This is a special case of corresponding results for general Hankel multipliers. Besides $L^p - L^q$ bounds we also characterize weak type inequalities and intermediate inequalities involving Lorentz spaces. Applications include results on interpolation of multiplier spaces.

1. Introduction

The purpose of this paper is to study convolution operators with radial kernels acting on radial L^p functions in $\mathbb{R}^d$. We are interested in the boundedness properties of such operators on L^p_{rad}, the space of radial L^p functions. It turns out (perhaps surprisingly) that for a large range of p one can actually prove a characterization in terms of the convolution kernel. Moreover we also obtain characterizations for the weak type (p,p) inequality, or, more generally, results involving the interpolating Lorentz spaces $L^{p,\sigma}_{\mathrm{rad}}$ for $p \le \sigma \le \infty$. Here $L^{p,\sigma}_{\mathrm{rad}}$ denotes the subspace of radial functions of the Lorentz space $L^{p,\sigma}(\mathbb{R}^d)$. Recall that we have the strict inclusion $L^{p,\sigma_1} \subset L^{p,\sigma_2}$ for $\sigma_1 < \sigma_2$. The space $L^{p,\infty}_{\mathrm{rad}}$ is the usual weak type p space, and of course $L^{p,p}_{\mathrm{rad}} = L^p_{\mathrm{rad}}$.

Let $K \in \mathcal{S}'(\mathbb{R}^d)$ be a radial convolution kernel, and denote by $\mathcal{T}_K$ the convolution operator $f \mapsto \mathcal{T}_K f = K * f$. We shall always assume that the Fourier transform $\widehat{K}$ is locally integrable; this is a trivial necessary condition for L^p boundedness (and also for $L^p \to L^q$ boundedness with $q \le 2$). Now consider the scaled kernels

$$K_t = t^{-d} K(t^{-1} \cdot).$$

Note that estimates for $\mathcal{T}_K$ imply appropriately scaled estimates for $\mathcal{T}_{K_t}$, $t > 0$. Let Φ be any radial Schwartz function whose Fourier transform is compactly supported in $\mathbb{R}^d \setminus \{0\}$. By using dilation invariance and testing

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the convolution with K_t on Φ, we get a trivial necessary condition for $L^{p,1}_{\mathrm{rad}} \to L^{p,\sigma}$ boundedness of $\mathcal{T}_K$, namely that

$$\sup_{t>0} \|\Phi * K_t\|_{L^{p,\sigma}} < \infty. \tag{1.1}$$

Our main result is that (1.1) for a single nontrivial radial Φ is also sufficient for the convolution to map L^p_{rad} to $L^{p,\sigma}$.

Theorem 1.1. *Let K be radial and let $\mathcal{T}_K$ be the associated convolution operator. Suppose $d > 1$, $1 < p < \frac{2d}{d+1}$, and $p \le \sigma \le \infty$. Then the following statements are equivalent:*

(a) There is a radial Schwartz-function Φ (not identically zero) for which condition (1.1) *is satisfied.*

(b) $\mathcal{T}_K$ extends to a bounded operator mapping $L^{p,1}_{\mathrm{rad}}(\mathbb{R}^d)$ to $L^{p,\sigma}_{\mathrm{rad}}(\mathbb{R}^d)$.

(c) $\mathcal{T}_K$ extends to a bounded operator mapping $L^{p}_{\mathrm{rad}}(\mathbb{R}^d)$ to $L^{p,\sigma}_{\mathrm{rad}}(\mathbb{R}^d)$.

As a consequence one can show that if in addition $\widehat{K}$ is compactly supported away from the origin then the L^p boundedness of $\mathcal{T}_K$ is equivalent with $K \in L^p_{\mathrm{rad}}$. *Cf.* §10 for this and somewhat stronger results for boundedness on Lorentz spaces. We remark that the condition $p < 2d/(d+1)$ is necessary since for $p \ge 2d/(d+1)$ there are radial L^p kernels whose Fourier transforms are unbounded and compactly supported in $\mathbb{R}^d \setminus \{0\}$, *cf.* the comment following Corollary 1.5 below.

It is convenient to formulate these characterizations for more general Fourier-Bessel (or Hankel) transforms of functions in $\mathbb{R}^+$. As it is well known ([33], ch. IV) the Fourier transform of radial functions can be expressed in terms of integral transforms on functions defined on $\mathbb{R}^+$, which is equipped with the measure $r^{d-1}dr$. To be specific we define the Fourier transform of a Schwartz function g in $\mathbb{R}^d$ by $\widehat{g}(\xi) \equiv \mathcal{F}_{\mathbb{R}^d}[g](\xi) = \int g(y) e^{-i\langle y,\xi\rangle} dy$. We recall that if g is *radial*, $g(x) = f(|x|)$ then its Fourier transform is radial and is given by

$$\widehat{g}(\xi) = (2\pi)^{d/2} \mathcal{B}_d f(\rho), \quad |\xi| = \rho, \tag{1.2}$$

where $\mathcal{B}_d$ denotes a Fourier-Bessel transform acting on functions on the half line. This transform can be defined for all *real* parameters $d > 1$, and it is given by

$$\mathcal{B}_d f(\rho) = \int_0^\infty f(s) B_d(s\rho) s^{d-1} ds \tag{1.3}$$

where

$$B_d(\rho) = \rho^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(\rho) \tag{1.4}$$

and J_α denotes the standard Bessel function. This definition is closely related with the classical (or *nonmodified*) Hankel transform given by

$$\mathcal{H}_\alpha f(x) = \int_0^\infty \sqrt{xy} J_\alpha(xy) f(y) dy;$$

indeed $\mathcal{B}_d = M_{-\frac{d-1}{2}} \mathcal{H}_{\frac{d-2}{2}} M_{\frac{d-1}{2}}$ where the multiplication operator M_c is defined by $M_c f(r) := r^c f(r)$. The operator $\mathcal{B}_d$ is just the *modified Hankel transform* $H_\nu \equiv H_\nu^{\mathrm{mod}}$ used in most papers on the subject, with the reparametrization $H_\nu^{\mathrm{mod}} = \mathcal{B}_{2\nu+2}$. We prefer our notation only because of the connection with radial Fourier multipliers when d is an integer. For $d = 1$ one recovers the cosine transform. If $d > 1$ is an integer then the function B_d in (1.4) represents (up to a constant) the Fourier transform of the surface measure on the unit sphere in $\mathbb{R}^d$. For general $d \geq 1$ the functions B_d are eigenfunctions with respect to the second order Bessel differential operator $L = -D^2 - \frac{d-1}{\rho} D$; here $D = d/d\rho$.

In what follows let

$$d\mu_d = r^{d-1} dr \tag{1.5}$$

and let $L^p(\mu_d)$ be the Lebesgue space of measurable functions f with

$$\|f\|_{L^p(\mu_d)} = \Big(\int_0^\infty |f(r)|^p r^{d-1} dr \Big)^{1/p} < \infty.$$

We continue to use the notation $\|f\|_p$ for the standard L^p norm on $\mathbb{R}$ (with respect to Lebesgue measure). Let $\mathcal{S}(\mathbb{R}_+)$ be the space of (restrictions to $\mathbb{R}^+$ of) even C^∞ functions on $\mathbb{R}$ for which all derivatives decrease rapidly; then $\mathcal{B}_d$ is an isomorphism of $\mathcal{S}(\mathbb{R}_+)$, an isometry of $L^2(\mathbb{R}_+, \mu_d)$, and $\mathcal{B}_d = \mathcal{B}_d^{-1}$. Clearly the space $\mathcal{S}(\mathbb{R}_+)$ is dense in $L^p(\mu_d)$. It is also useful to note that the space $\mathcal{B}_d(C_0^\infty)$ is dense in $L^p(\mu_d)$ for $1 < p < \infty$; here C_0^∞ is the class of C^∞ functions with compact support in $(0, \infty)$. This statement is proved in Theorem 4.7 of [36]. Clearly, if m is locally integrable on $\mathbb{R}^+$ the operator T_m defined by

$$T_m f(r) = \mathcal{B}_d[m \mathcal{B}_d f](r) \tag{1.6}$$

is well defined for $f \in \mathcal{B}_d(C_0^\infty)$. We remark that $L^1(\mu_d)$ is a commutative Banach algebra with respect to a certain convolution structure [17], and the operators (1.6) can then be regarded as generalized convolutions. However in this paper we shall not need to make use of the precise definition of the convolution structure.

We now formulate necessary and sufficient characterizations for $L^p \to L^q$ boundedness for T_m as well as extensions to Lorentz space inequalities. Our main characterization is in terms of size properties of the one-dimensional Fourier transform of localizations of m.

Theorem 1.2. *Let $m \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and let ϕ be a C^∞ function compactly supported in $\mathbb{R}_+$ (not identically zero). Suppose $1 < d < \infty$, $1 < p < \frac{2d}{d+1}$, $p \le q < 2$ and $p \le \sigma \le \infty$. Then the following statements are equivalent.*

(i) T_m extends to a bounded operator $T_m : L^{p,\omega}(\mu_d) \to L^{q,\sigma}(\mu_d)$, for $\omega = \min\{\sigma, q\}$.

(ii) T_m extends to a bounded operator $T_m : L^{p,1}(\mu_d) \to L^{q,\sigma}(\mu_d)$.

(iii)

$$\sup_{t>0} t^{d(\frac{1}{p}-\frac{1}{q})} \big\| \mathcal{B}_d[\phi m(t\cdot)] \big\|_{L^{q,\sigma}(\mu_d)} < \infty. \tag{1.7}$$

(iv) With $k_t(x) = \mathcal{F}^{-1}_{\mathbb{R}}[\phi m(t\cdot)](x)$, the condition

$$\sup_{t>0} t^{d(\frac{1}{p}-\frac{1}{q})} \big\| (1+|\cdot|)^{-\frac{d-1}{2}} k_t \big\|_{L^{q,\sigma}((1+|x|)^{d-1}dx)} < \infty \tag{1.8}$$

holds.

Condition (1.8) is simpler when $q = \sigma$, and in this case we see that T_m is bounded from $L^p(\mu_d)$ to $L^q(\mu_d)$ (and in fact from $L^{p,q}(\mu_d)$ to $L^q(\mu_d)$) if and only if

$$\sup_{t>0} t^{d(\frac{1}{p}-\frac{1}{q})} \Big(\int_{-\infty}^{\infty} \big|k_t(x)\big|^q (1+|x|)^{(d-1)(1-\frac{q}{2})} dx \Big)^{\frac{1}{q}} < \infty; \tag{1.9}$$

here again $1 < p < \frac{2d}{d+1}$ and now $p \le q \le 2$ (for the case $q = 2$ see §8).

Theorem 1.1 is an immediate consequence of Theorem 1.2. If $K = \mathcal{F}^{-1}_{\mathbb{R}^d}[m(|\cdot|)]$ and $g(x) = f(|x|)$ then $\mathcal{T}_K g(x) = T_m f(|x|)$, by (1.2), and the condition (1.7) is equivalent with

$$\sup_{t>0} t^{d(1/p-1/q)} \big\| \mathcal{F}^{-1}_{\mathbb{R}^d}[\phi(|\cdot|) m(t|\cdot|)] \big\|_{L^{q,\sigma}(\mathbb{R}^d)} < \infty. \tag{1.10}$$

Alternatively, after rescaling, one can express this condition using the homogeneous Besov type space $\dot{B}_{-d/p',\infty}(L^{q,\sigma})$. Namely for radial K (with $\widehat{K} \in L^1_{\mathrm{loc}}$) and $\Phi_t := \mathcal{F}^{-1}[\phi(t|\cdot|)]$,

$$\begin{aligned} \big\| \mathcal{T}_K \big\|_{L^p_{\mathrm{rad}}(\mathbb{R}^d) \to L^{q,\sigma}_{\mathrm{rad}}(\mathbb{R}^d)} &\approx \|K\|_{\dot{B}_{-d/p',\infty}(L^{q,\sigma})} \\ &\approx \sup_{t>0} t^{-d/p'} \big\| \Phi_{1/t} * K \big\|_{L^{q,\sigma}}. \end{aligned} \tag{1.11}$$

Note that the expression on the right hand side becomes a norm only after considering the quotient of the space of distributions modulo polynomials; however the (necessary) assumption that $\widehat{K}$ is locally integrable excludes polynomials (and even nonzero constants). As a special case (using the more

familiar notation when $q=\sigma$) the operator $\mathcal{T}_K$ maps boundedly $L^p_{\mathrm{rad}}(\mathbb{R}^d) \to L^q_{\mathrm{rad}}(\mathbb{R}^d)$ if and only if $\widehat{K}$ is locally integrable and $K \in \dot{B}^q_{-d/p',\infty}$.

We remark that no characterizations for $p \neq 1$ seem to have been observed before; however almost sharp results on compactly supported multipliers on $L^p_{\mathrm{rad}}(L^2_{\mathrm{sph}})$ spaces on $\mathbb{R}^d$, are in [22], in the sense that the exponent $(d-1)(1-p/2)$ in (1.9) is replaced by $(d-1)(1-p/2)+\varepsilon$. Arai [1] proved a similar result with ε-loss for global Hankel multipliers, essentially by combining arguments in [22] and [28]. We also note that the necessity of the condition (1.7) is trivial, and the necessity of conditions related to (1.8) is known from [16], [27], and [1]; *cf.* also §4 for an elementary proof of the implication $(iii) \implies (iv)$ in Theorem 1.2. Finally note that Theorem 1.2 can be combined with transplantation theorems for nonmodified Hankel transforms ([17], [36], [35], [24]) to derive results on some other weighted L^p spaces.

We state two consequences of the above characterizations concerning the structure of multiplier spaces. It is convenient to define $\mathfrak{M}^{p,q}_d$, for $1<p\le q\le 2$ as the space of all locally integrable functions m for which T_m extends to a bounded operator from $L^p(\mu_d)$ to $L^q(\mu_d)$, and the norm is given by the operator norm of T_m.

A first implication of Theorem 1.2 is that local multiplier conditions imply global ones; we state the case for $p=q$. Namely for nontrivial $\phi \in C_c^\infty(\mathbb{R}_+)$ one has the following equivalence.

Corollary 1.3. *For $d>1$, $1<p<\frac{2d}{d+1}$,*

$$\|m\|_{\mathfrak{M}^{p,p}_d} \approx \sup_{t>0} \|\phi m(t\cdot)\|_{\mathfrak{M}^{p,p}_d}. \tag{1.12}$$

It is well known that the analogue of this corollary for $d=1$ and even classes of continuous Fourier multipliers in M^p on the real line is false, see examples by Littman, McCarthy and Rivière [21] and by Stein and Zygmund [34].

Another failing analogy to $M^p(\mathbb{R})$ concerns the subject of interpolation. As a straightforward consequence of the characterization we obtain an interpolation result with respect to the second complex interpolation method $[\cdot,\cdot]^\theta$, introduced by Calderón (see [4], and [2], p.88). In contrast, an extension of a result of Zafran ([38]), states that the space $M^p(\mathbb{R})$, $1<p<2$, is not an interpolation space for any pair (M^{p_0}, M^{p_1}) with $p_0<p<p_1$, see Appendix §A.

Corollary 1.4. *Suppose $1<d_i<\infty$, $1<p_i<\frac{2d_i}{d_i+1}$, $p_i\le q_i\le 2$, for $i=0,1$, moreover that* $(1/p,1/q,d)=(1-\vartheta)(1/p_0,1/q_0,d_0)+\vartheta(1/p_1,1/q_1,d_1)$

with $0<\vartheta<1$. *Then*

$$[\mathfrak{M}^{p_0,q_0}_{d_0},\mathfrak{M}^{p_1,q_1}_{d_1}]^{\vartheta}=\mathfrak{M}^{p,q}_{d}. \tag{1.13}$$

This result follows from interpolation of certain Fourier-localized versions of weighted L^p spaces (which are defined by (1.8)), see Lemma 2.5 below. For a related result on real interpolation see §10.

Finally by standard arguments using Hölder's inequality and Plancherel's theorem condition (1.8) implies the known sufficient criteria of Hörmander type ([14]), which are formulated using localized L^2-Sobolev spaces; these were termed $S(2,\alpha)$ in [7] and $WBV_{2,\alpha}$ (with $\alpha>1/2$) in [15]. The following endpoint bounds in terms of localized versions of Besov spaces seem to be new; it is an optimal estimate within the class of L^2-smoothness assumptions. Recall $\|g\|_{B^2_{a,q}}\approx(\sum_{k=0}^{\infty}2^{kaq}\|\widehat{g}\|^q_{L^2(\mathcal{I}_k)})^{1/q}$ where $\mathcal{I}_0=[-1,1]$ and $\mathcal{I}_k=\{\xi\in\mathbb{R}:2^{k-1}\le|\xi|\le 2^k\}$, for $k>1$.

Corollary 1.5. *For* $1<d<\infty$, $1<p<\frac{2d}{d+1}$, $p\le q\le 2$,

$$\|m\|_{\mathfrak{M}^{p,q}_d}\lesssim\sup_{t>0}t^{d(\frac1p-\frac1q)}\|\phi m(t\cdot)\|_{B^2_{a,q}},\quad a=d(\frac1q-\frac12). \tag{1.14}$$

Here, and in what follows, the notation $\lesssim$ indicates that in the inequality an unspecified constant is involved which may depend on d,p,q. Since the space $B^2_{1/2,p}$ contains unbounded functions for $p>1$ the corollary does not extend to the endpoint $p=q=2d/(d+1)$.

This paper. In §2 we gather various facts on Bessel functions, Littlewood-Paley inequalities, interpolation and elementary convolution inequalities on weighted spaces, needed later in the paper. In §3 we derive some pointwise bounds for the kernels of multiplier transformations, assuming that the multipliers are compactly supported in $(1/2,2)$. In §4 we prove the necessity of the conditions, namely the implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) of Theorem 1.2. The proof of the main implication (iv) $\implies$ (i) is contained in sections 5-9. In §5 we discuss the basic decomposition into Hardy type and singular integral operators. The crucial estimate for the main Hardy-type operator is proved in §6, and §7 contains estimates for better behaved operators (in particular singular integrals) for which we do not need the full strength of assumption (1.8). In §8 we give the straightforward proof of the $L^p\to L^2$ bounds and then conclude in §9 the proof of the implication (iv) $\implies$ (i) by an interpolation. In §10 we give the short proofs of the Corollaries and briefly discuss a further result on real interpolation and an improved version of our results for multipliers which are compactly supported away from the origin. Some open problems are mentioned in §11. An appendix (§A) is included with the above mentioned non-interpolation results for Fourier multipliers.

2. Preliminaries

Asymptotics for Bessel functions. In order to relate the Hankel transforms of multipliers to the one-dimensional Fourier transform we need to use standard asymptotics for Bessel functions (see [10], 7.13.1(3)), namely for $|x| \ge 1$,

$$B_d(x) = \sum_{\nu=0}^{M} c_{\nu,d} \cos(x - \tfrac{d-1}{4}\pi) x^{-2\nu - \frac{d-1}{2}} \\ + \sum_{\nu=0}^{M} \widetilde{c}_{\nu,d} \sin(x - \tfrac{d-1}{4}\pi) x^{-2\nu - \frac{d+1}{2}} + x^{-M} \widetilde{E}_{M,d}(x)$$

with $c_{0,d} = (2/\pi)^{1/2}$, and the derivatives of $\widetilde{E}_{M,d}$ are bounded. Thus one may also write down expansions for the derivatives and, after writing the cosine and sine terms as combinations of exponentials and applying the previous formula with M replaced by $M+k$ one also gets, for $|x| \ge 1$,

$$B_d^{(k)}(x) = \sum_{\nu=0}^{M} (c^{+}_{\nu,k,d} e^{ix} + c^{-}_{\nu,k,d} e^{-ix}) x^{-\nu - \frac{d-1}{2}} + x^{-M} E_{M,k,d}(x) \tag{2.1}$$

where $c^{\pm}_{0,0,d} = (2\pi)^{-1/2} e^{\mp i \frac{d-1}{4}\pi}$ and the $E_{M,k,d}$ have bounded derivatives:

$$|E^{(k_1)}_{M,k,d}(x)| \le C(M,k,k_1,d). \tag{2.2}$$

Littlewood-Paley inequalities. Let $\eta \in C^\infty(\mathbb{R}_+)$ with compact support away from 0. Let $L_j f = \mathcal{B}_d[\eta(2^{-j}\cdot)\mathcal{B}_d f]$. Then for $1 < p < \infty$ there are the inequalities

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |L_j f|^2 \Big)^{1/2} \Big\|_{L^p(\mu_d)} \le C_p \|f\|_{L^p(\mu_d)}, \tag{2.3}$$

$$\Big\| \sum_{j\in\mathbb{Z}} L_j f_j \Big\|_{L^p(\mu_d)} \le C'_p \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^2 \Big)^{1/2} \Big\|_{L^p(\mu_d)}; \tag{2.4}$$

indeed (2.3) and (2.4) are dual to each other with $C'_p = C_{p'}$, $1/p + 1/p' = 1$. By the real (Lions-Peetre) interpolation method the spaces $L^p(\mu_d)$ can be replaced by $L^{p,\sigma}(\mu_d)$, for any σ.

For the proof of (2.3), (2.4) we note that the operators

$$f \mapsto \sum_j \pm L_j f$$

are bounded on $L^p(\mu_d)$, $1 < p < \infty$, with operator norm independent of the choice of signs $\pm$. This follows for example by (a non-sharp version of) the Hörmander type multiplier criterion for modified Hankel transforms in

Gasper and Trebels [14]; for the case of integer d one could simply use standard results in $\mathbb{R}^d$ specialized to radial functions ([31]). Now the inequalities (2.3), (2.4) follow by the usual averaging argument using Rademacher functions (see [31], ch. IV, §5.2), and a duality argument.

Remarks on Lorentz spaces. We assume that Ω is a measure space with given σ-algebra and underlying measure μ. We refer to a thorough discussion of Lorentz spaces to [33]. There the definition of $L^{q,\sigma}$ is given in terms of rearrangements of f and it is shown that this definition is equivalent to a norm when $1 < q < \infty$, $1 \le \sigma \le \infty$. Instead of the rearrangement function one can also use the distribution function and it is easy to check (on simple functions) that an equivalent quasi-norm on $L^{q,\sigma}$ is given by

$$\|f\|_{L^{q,\sigma}} \approx \Big(\sum_{\ell=-\infty}^{\infty} 2^{\ell\sigma} \big[\mu\big(\{x \in \Omega : |f(x)| > 2^\ell\}\big)\big]^{\sigma/q}\Big)^{1/\sigma} \tag{2.5}$$

(with the natural ℓ^∞ analogue for $\sigma = \infty$). For the manipulation of vector-valued functions we shall need the following inequality.

Lemma 2.1. *Let $1 < q < r$, $1 \le \sigma \le \infty$ and let $\{F_j\}$ be a sequence of measurable functions on Ω. Then*

$$\Big\| \Big(\sum_j |F_j|^r \Big)^{1/r} \Big\|_{L^{q,\sigma}} \le C(q, \sigma, r) \Big(\sum_j \|F_j\|_{L^{q,\sigma}}^\omega \Big)^{1/\omega}, \quad \omega = \min\{\sigma, q\}. \tag{2.6}$$

Proof. Consider measurable functions H on $\Omega \times \mathbb{Z}$. We first claim that for $1 < q < r$, $1 \le \sigma \le \infty$

$$\Big\| \Big(\sum_j |H(\cdot, j)|^r \Big)^{1/r} \Big\|_{L^{q,\sigma}(\Omega)} \le c(q, \sigma, r) \|H\|_{L^{q,\sigma}(\Omega \times \mathbb{Z})}. \tag{2.7}$$

For the case $q = \sigma$ this follows by applying the imbedding $\ell^q \hookrightarrow \ell^r$ and then Fubini's theorem (interchanging a sum and an integral). For arbitrary σ it follows by applying the real method of interpolation. Now we apply (2.5) to the right hand side of (2.7) and estimate for $\sigma \ge q$

$$\|H\|_{L^{q,\sigma}(\Omega\times\mathbb{Z})} \lesssim \Big(\sum_\ell 2^{\ell\sigma} \Big(\sum_j \mu\big(\{x : |H(x, j)| > 2^\ell\}\big)\Big)^{\sigma/q}\Big)^{1/\sigma} \lesssim$$

$$\Big(\sum_j \Big(\sum_\ell 2^{\ell\sigma} \mu\big(\{x : |H(x, j)| > 2^\ell\}\big)^{\sigma/q}\Big)^{q/\sigma}\Big)^{1/q} \lesssim \Big(\sum_j \big\|H(\cdot, j)\big\|_{L^{q,\sigma}}^q\Big)^{1/q};$$

here we have used Minkowski's inequality for the sequence space $\ell^{\sigma/q}$. If $\sigma < q$ we use instead the imbedding $\ell^{\sigma/q} \subset \ell^1$ and estimate $\|H\|_{L^{q,\sigma}(\Omega\times\mathbb{Z})}$ by

$$\Big(\sum_\ell 2^{\ell\sigma} \sum_j \big(\mu\big(\{x : |H(x, j)| > 2^\ell\}\big)\big)^{\sigma/q}\Big)^{1/\sigma} \approx \Big(\sum_j \big\|H(\cdot, j)\big\|_{L^{q,\sigma}}^\sigma\Big)^{1/\sigma}.$$

□

Elementary inequalities for weighted norms. To handle expressions such as (1.8) we need some elementary inequalities on convolutions and dilations.

Lemma 2.2. *Let $a \ge 0$, and $\gamma > a+1$. Suppose that g, ζ are Lebesgue measurable on $\mathbb{R}$ and ζ satisfies*

$$|\zeta(x)| \le C_1(1+|x|)^{-\gamma}. \tag{2.8}$$

Then for $q_1 \ge q \ge 1$

$$\Big(\int |g*\zeta(x)|^{q_1}(1+|x|)^{aq_1}dx\Big)^{1/q_1} \lesssim C_1\Big(\int |g(x)|^q(1+|x|)^{aq}dx\Big)^{1/q}. \tag{2.9}$$

Also

$$\Big(\int |g(tx)|^q(1+|x|)^{aq}dx\Big)^{1/q} \le t^{-1/q}\max\{1,t^{-a}\}\Big(\int |g(x)|^q(1+|x|)^{aq}dx\Big)^{1/q}. \tag{2.10}$$

Proof. For $q=q_1$ the left hand side of (2.9) is dominated by a constant times

$$\int(1+|y|)^{-\gamma}\Big(\int |g(x-y)|^q(1+|x|)^{aq}dx\Big)^{1/q}dy$$
$$\le \int(1+|y|)^{-\gamma+a}dy\,\Big(\int |g(x)|^q(1+|x|)^{aq}dx\Big)^{1/q}$$

where we have used $1+|x| \le (1+|x-y|)(1+|y|)$. The integral is finite since $\gamma > a+1$.

The analogue of (2.9) for $q_1=\infty$ is also valid; we estimate (assuming momentarily $q>1$)

$$|g*\zeta(x)|(1+|x|)^a \lesssim$$
$$\Big(\int |g(x-y)(1+|x-y|)^a|^q dy\Big)^{1/q}\Big(\int \frac{(1+|y|)^{-\gamma q'}(1+|x|)^{aq'}}{(1+|x-y|)^{aq'}}dy\Big)^{1/q'}$$

where the first term is the desired expression on the right hand side of (2.9) and the second term is $\lesssim (\int(1+|y|)^{(a-\gamma)q'}dy)^{1/q'}$, hence finite. A similar argument holds for $q=1$. We have now proved the asserted bound for $q_1=\infty$ and $q_1=q$ and the intermediate cases follow by interpolation.

Inequality (2.10) follows from $(1+|x|/t) \le \max\{t^{-1},1\}(1+|x|)$ and a change of variable. □

We shall need the following Lorentz space variant of Lemma 2.2 which will be used repeatedly.

Lemma 2.3. *Let $\alpha > \beta q \ge 0$, $1 < q < \infty$, $1 \le \sigma \le \infty$, and let $d\nu_\alpha = (1+|x|)^\alpha dx$ (as a measure on $\mathbb{R}$). Suppose that ζ satisfies* (2.8) *for some $\gamma > 1-\beta+\alpha/q$. Then*

$$\Big\|\frac{g*\zeta}{(1+|\cdot|)^\beta}\Big\|_{L^{q,\sigma}(\nu_\alpha)} \lesssim \Big\|\frac{g}{(1+|\cdot|)^\beta}\Big\|_{L^{q,\sigma}(\nu_\alpha)} \tag{2.11}$$

and

$$\Big\|\frac{g(t\cdot)}{(1+|\cdot|)^\beta}\Big\|_{L^{q,\sigma}(\nu_\alpha)} \lesssim t^{-1/q}\max\{1,t^{-\alpha/q+\beta}\Big\|\frac{g}{(1+|\cdot|)^\beta}\Big\|_{L^{q,\sigma}(\nu_\alpha)} \tag{2.12}$$

Proof. Define $\mathcal{M}_\beta f := (1+|x|)^\beta f(x)$ and let $S_\zeta f(x) = \zeta * f$. Then the assertion is equivalent with the claim that $\mathcal{M}_{-\beta} S_\zeta \mathcal{M}_\beta$ is bounded on $L^{q,\sigma}(\nu_\alpha)$. Since $1 < q < \infty$ and restriction on γ also involves a strict inequality the general Lorentz space estimate follows from the case $q = \sigma$ by real interpolation. The $L^q(\nu_\alpha)$ boundedness of $\mathcal{M}_{-\beta} S_\zeta \mathcal{M}_\beta$ is in turn equivalent to the inequality (2.9) for the choice $q = q_1$ and $aq = \alpha - \beta q$. We may apply (2.9) since $\gamma > a+1 = 1-\beta+\alpha/q$. The proof of (2.12) is similar. $\square$

Independence of the localizing function. Let $a \ge 0$, $b \in \mathbb{R}$, $1 \le p \le 2$. Let ϕ be a smooth function supported on a compact subinterval of $(0,\infty)$, and assume that ϕ is not identically zero. It will be convenient to denote by $\mathrm{LF}(p,a,b)$ the space of all m which are integrable over every compact subinterval of $(0,\infty)$ and satisfy the condition

$$\sup_{t>0} t^b\Big(\int_{-\infty}^{\infty} \big|\mathcal{F}_{\mathbb{R}}^{-1}[\phi m(t\cdot)](x)\big|^p (1+|x|)^{ap} dx\Big)^{1/p} \le A \tag{2.13}$$

for some finite A. Here LF refers to localization and to the Fourier transform.

We use Lemma 2.2 and Lemma 2.3 to prove that the choice of the cutoff function ϕ in (2.13) and (1.8) does not matter. Moreover we wish, for suitable ϕ, use discrete conditions where the sup is taken over dyadic t. To formulate these choose $\varphi \in C_c^\infty(\frac12, 2)$ with the property that

$$\sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}s) = 1, \quad s > 0. \tag{2.14}$$

Lemma 2.4. *Let $1 < q < \infty$, $1 \le \sigma \le \infty$.*

(i) Suppose

$$\sup_{t>0} t^b \Big\|\frac{|\mathcal{F}_{\mathbb{R}}^{-1}[\phi m(t\cdot)]}{(1+|\cdot|)^\beta}\Big\|_{L^{q,\sigma}(\nu_\alpha)} \le A < \infty \tag{2.15}$$

holds for some $\phi \in C_c^\infty(\mathbb{R}^+)$ which is not identically zero. Let $\eta \in C_c^\infty(\mathbb{R}^+)$. Then the expression analogous to (2.13)*, but with ϕ replaced by η, is bounded by CA, where C does not depend on m.*

(ii) With $\varphi \in C_c^\infty(\frac12, 2)$ satisfying (2.14) the left hand side of (2.15) is bounded by

$$C \sup_j 2^{jb} \Big\| \frac{|\mathcal{F}_{\mathbb{R}}^{-1}[\varphi m(2^j \cdot)]|}{(1+|\cdot|)^\beta} \Big\|_{L^{q,\sigma}(\nu_\alpha)}. \tag{2.16}$$

Proof. We begin by observing that $\int_0^\infty \phi^2(\tau s)\frac{d\tau}{\tau} = c > 0$ independent of s. Hence $\eta(s)m(ts) = c^{-1}\int_0^\infty \phi^2(\tau s)\eta(s)m(ts)\tau^{-1}d\tau$ and since if s is taken from a compact subset of $(0,\infty)$ the integral reduces to an integral over $[\varepsilon, \varepsilon^{-1}]$ for some $\varepsilon \in (0,1)$. Thus

$$\mathcal{F}^{-1}[\eta(s)m(ts)] = \int_\varepsilon^{1/\varepsilon} \Phi_\tau * \big(\tau^{-1}k_{t/\tau}(\tau^{-1}\cdot)\big)\frac{d\tau}{\tau}$$

where $\Phi_\tau = \mathcal{F}^{-1}[\phi(\tau\cdot)\eta]$ and $k_t = \mathcal{F}^{-1}[\phi m(t\cdot)]$. Now the assertion (i) follows immediately from (2.11) and (2.12). (ii) is proved similarly; the details are left to the reader. □

Interpolation. Interpolation results for the spaces $\mathrm{LF}(p,a,b)$ are analogous to those for localized potential spaces in [7], [5], with a very similar proof; therefore we only give a sketch. We denote by $[\cdot,\cdot]_\vartheta$, $[\cdot,\cdot]^\vartheta$ the complex interpolation methods of Calderón (see [4], and also ch. 4 in [2]).

Lemma 2.5. *Let $a_0, a_1 \ge 0$, $b_0, b_1 \in \mathbb{R}$ and $1 \le p_0, p_1, \le 2$. Suppose that $(a, b, p^{-1}) = (1-\vartheta)(a_0, b_0, p_0^{-1}) + \vartheta(a_1, b_1, p_1^{-1})$ and $0 < \vartheta < 1$. Then*

$$\big[\mathrm{LF}(p_0, a_0, b_0), \mathrm{LF}(p_1, a_1, b_1)\big]^\vartheta = \mathrm{LF}(p, a, b). \tag{2.17}$$

Sketch of proof. Let $\|K\|_{L(p,a)} := (\int_{-\infty}^\infty |K(t)|^p(1+|t|)^{ap}dt)^{1/p}$ and denote by $\ell_b^\infty(L(p,a))$ be the space of sequences of $L(p,a)$ functions $\{G_j\}_{j\in\mathbb{Z}}$ for which $\sup_j 2^{jb}\|G_j\|_{L(p,a)} < \infty$. Weighted L^p spaces can be interpolated by the complex method (see [2], ch. 5) and we have

$$L(p,a) = [L(p_0,a_0), L(p_1,a_1)]_\vartheta.$$

By a result of Calderón ([4], §13.6)

$$\ell_b^\infty(L(p,a)) = [\ell_{b_0}^\infty(L(p_0,a_0)), \ell_{b_1}^\infty(L(p_1,a_1))]^\vartheta \tag{2.18}$$

and one has to show that $\mathrm{LF}(p,a,b)$ is a retract of $\ell_b^\infty(L(p,a))$; *i.e.* there are bounded linear operators

$$\mathfrak{A} : \mathrm{LF}(p,a,b) \to \ell_b^\infty(L(p,a)),$$
$$\mathfrak{B} : \ell_b^\infty(L(p,a)) \to \mathrm{LF}(p,a,b),$$

so that $\mathfrak{B} \circ \mathfrak{A}$ is the identity on $\mathrm{LF}(p, a, b)$. These maps are given by

$$[\mathfrak{A}m]_j = \mathcal{F}^{-1}[\varphi m(2^j \cdot)], \tag{2.19}$$

$$\mathfrak{B}G = \sum_{k\in\mathbb{Z}} \varphi(2^{-k}\cdot)\widehat{G_k}(2^{-k}\cdot). \tag{2.20}$$

$\mathfrak{A}$ is bounded by definition of the $\mathrm{LF}(p, a, b)$ norm and the boundedness of $\mathfrak{B}$ is straightforward; one uses Lemma 2.2. Also $\mathfrak{B} \circ \mathfrak{A}$ is the identity on $\mathrm{LF}(a, b, p)$, by (2.14). This shows (2.17), the details are left to the reader. □

Remark. The analogues of these theorems for localized potential spaces are proved by Connett and Schwartz in [7], see also [5]. In [7] it is also noted that the analogue of (2.17) fails for the $[\cdot,\cdot]_\vartheta$ method (and their argument applies here as well). In addition, if $\mathrm{LF}_o(p, a, b)$ denotes the closed subspace of functions for which the expressions $2^{jb}\|\mathcal{F}_{\mathbb{R}}^{-1}[\varphi m(2^j\cdot)]\|_{L^p((1+|x|)^{ap}dx)}$ tend to 0 as $|j| \to \infty$, then one also has $\big[\mathrm{LF}_o(p_0, a_0, b_0), \mathrm{LF}_o(p_1, a_1, b_1)\big]_\vartheta = \mathrm{LF}_o(p, a, b)$. This is analogous to a result in [7] on localized potential spaces.

3. Kernel estimates

Assume that the multiplier m has compact support in $[\frac12, 2]$. Here we give pointwise estimates for the kernel of multiplier transformations involving two Bessel transforms $\mathcal{B}_a$, $\mathcal{B}_b$ of possibly different orders; however the main interesting case is of course $a = b = d$. We can write for $a, b > 0$

$$\mathcal{B}_a[m\mathcal{B}_b f](r) = \int \mathcal{K}_{a,b}[m](r, s)s^{b-1}f(s)ds \tag{3.1}$$

where the kernel is given by

$$\mathcal{K}_{a,b}(r, s) \equiv \mathcal{K}_{a,b}[m](r, s) = \int_0^\infty m(\rho)B_a(\rho r)B_b(\rho s)\rho^{a-1}d\rho. \tag{3.2}$$

Proposition 3.1. *Let $a \ge 1$, $b \ge 1$, $N > 1$ and let m be integrable and supported in $[\frac12, 2]$. Then for $\beta, \gamma = 0, 1, 2, \dots$*

$$\begin{aligned} &\big|\partial_r^\beta \partial_s^\gamma \mathcal{K}_{a,b}[m](r, s)\big| \le \\ &\qquad C_{N,\beta,\gamma} \sum_{(\pm,\pm)} (1 + r)^{-\frac{a-1}{2}}(1 + s)^{-\frac{b-1}{2}} \int \frac{|\mathcal{F}_{\mathbb{R}}^{-1}[m](\pm r \pm s - u)|}{(1 + |u|)^N} du. \end{aligned} \tag{3.3}$$

Proof. We begin with a preliminary observation, which we shall use several times, namely the inequality

$$(1 + R)^{-M} \int \frac{|g(u)|}{(1 + |u|)^{N_1}} du \le C(1 + R)^{-M+N_1} \int \frac{|g(R + u)|}{(1 + |u|)^{N_1}} du; \tag{3.4}$$

this is (similar to the statement in Lemma 2.2) a consequence of the triangle inequality and a translation in the integral.

Let χ be a C^∞ function so that $\chi(s) = 1$ for $s \in (1/2, 2)$ and χ is supported in $(1/4, 4)$. If $r, s \le 1$ then the function

$$\rho \mapsto h(\rho) = \chi(\rho)\rho^{a-1+\beta+\gamma} B_a^{(\beta)}(\rho r) B_b^{(\gamma)}(\rho s)$$

is smooth and has a rapidly decaying Fourier transform, with bounds uniform in $r, s \le 1$. Denote the Fourier transform by $u \mapsto \lambda(r, s, u)$. We may apply duality for the Fourier transform and estimate (with $\kappa = \mathcal{F}_{\mathbb{R}}^{-1}[m]$)
(3.5)

$$|\partial_r^\beta \partial_s^\gamma \mathcal{K}_{a,b}[m](r,s)| = \Big| \int \kappa(u)\lambda(r,s,u)du \Big| \le C_{N_1,\beta,\gamma} \int \frac{|\kappa(u)|}{(1+|u|)^{N_1}} du.$$

Clearly this term is bounded by a suitable constant times any of the terms on the right hand side of (3.3), as long as $r, s \le 1$.

Next we consider the case $s \le 1$, $r \ge 1/2$ and use the asymptotic expansion (2.1) for $B_a(\rho r)$ and its derivatives. We assume that the parameter M is chosen large, in order to use (3.4), in fact we require $M > 2N + (a+b)/2$.

This yields

$$\partial_r^\beta \partial_s^\gamma \mathcal{K}_{a,b}(r,s) = \sum_{\pm} \sum_{\nu=0}^{M} r^{-\frac{a-1}{2}-\nu} \int m(\rho) e^{\pm ir\rho} \eta^{\pm}_{\nu,\beta,a,b}(s,\rho) d\rho \\ + r^{-M} \int m(\rho) \omega_{M,\beta,\gamma,a,b}(r,s,\rho) d\rho$$

where

$$\eta^{\pm}_{\nu,\beta,a,b}(s,\rho) = c^{\pm}_{\nu,\beta,a} \chi(\rho) \rho^{\frac{a-1}{2}+\beta+\gamma-\nu} B_b^{(\gamma)}(s\rho)\,,$$
$$\omega_{M,\beta,\gamma,a,b}(r,s,\rho) = \rho^{-M+\beta+\gamma+a-1} E_{M,\beta,a}(r\rho) B_b^{(\gamma)}(s\rho)\,.$$

The terms in the sum can be realized as convolutions of κ with rapidly decaying functions, multiplied with $r^{-\frac{a-1}{2}-\nu}$. These terms are bounded by

$$r^{-\frac{a-1}{2}} \int \frac{|\kappa(\mp r - u)|}{(1+|u|)^N} du$$

and since $s \lesssim 1$ this also implies the bound by the sum of terms on the right hand side of (3.3). For the error term we argue as above, using duality to estimate

$$r^{-M} \Big| \int \kappa(u) \widehat{\omega}_{M,\beta,\gamma,a,b}(r,s,u) du \Big| \lesssim r^{-M+N} \int \frac{|\kappa(u)|}{(1+|u|)^N} du$$

and the desired estimate follows from using (3.4), recall $M > 2N + (a-1)/2$.

The estimations for the case $r \lesssim 1$ and $s \gtrsim 1$ are similar, the roles of r and s are reversed.

Finally, to handle the case $r, s \geq 1/2$ we use the asymptotic expansion (2.1) for both $B_a(\rho s)$ and $B_b(\rho r)$, again with large M. We then write

$$\partial_r^\beta \partial_s^\gamma \mathcal{K}_{a,b}(r,s) = \int_0^\infty m(\rho)\rho^{a-1+\beta+\gamma} B_a^{(\beta)}(r\rho) B_b^{(\gamma)}(s\rho) d\rho$$

as

(3.6)

$$\begin{aligned}
&\sum_{\nu,\nu'}\sum_{\pm,\pm} c^{\pm}_{\nu,\beta,a} c^{\pm}_{\nu',\gamma,b} r^{-\frac{a-1}{2}-\nu} s^{-\frac{b-1}{2}-\nu'} \int m(\rho)\rho^{\frac{a-b}{2}+\beta+\gamma-\nu-\nu'} e^{i\rho(\pm r\pm s)} d\rho \\
&+ \sum_{\nu}\sum_{\pm} c^{\pm}_{\nu,\beta,a} r^{-\frac{a-1}{2}-\nu} s^{-M} \int m(\rho)\rho^{\frac{a-1}{2}+\beta+\gamma-\nu-M} E_{M,\gamma,b}(\rho s) e^{\pm i\rho r} d\rho \\
&+ \sum_{\nu'}\sum_{\pm} c^{\pm}_{\nu',\gamma,b} s^{-\frac{b-1}{2}-\nu'} r^{-M} \int m(\rho)\rho^{a-\frac{b+1}{2}+\beta+\gamma-\nu'-M} E_{M,\beta,a}(\rho r) e^{\pm i\rho s} d\rho \\
&+ (rs)^{-M} \int m(\rho)\rho^{a-1+\beta+\gamma-2M} E_{M,\beta,a}(\rho r) E_{M,\gamma,b}(\rho s) d\rho.
\end{aligned}$$

The first (double) sum in (3.6) is clearly bounded by the right hand side of (3.3). The second, third and fourth terms are bounded, by the previous arguments by a constant times

$$r^{-\frac{a-1}{2}} s^{N-M} \int \frac{|\kappa(\mp r+u)|}{(1+|u|)^N} du, \quad s^{-\frac{b-1}{2}} r^{N-M} \int \frac{|\kappa(\mp s+u)|}{(1+|u|)^N} du,$$

and $(rs)^{N-M} \int |\kappa(u)|(1+|u|)^{-N} du$, respectively. However by using inequality (3.4) and the condition $M > 2N + (a+b)/2$ these terms are seen to be also bounded by the right hand side of (3.3). □

Proposition 3.1 is mainly interesting as an estimate for general multipliers. However for the proof of necessary conditions we record a straightforward consequence for smooth multipliers, in the special case where $a = 1$, $b = d$.

Corollary 3.2. *Let $d \geq 1$ and let $\chi \in C^\infty$ be supported in $[1/4, 4]$. Then for any $M \geq 0$*

$$|\mathcal{B}_1[\chi \mathcal{B}_d f](r)| \leq C_M \int_0^\infty \frac{|f(s)| s^{d-1}}{(1+|r-s|)^M (1+s)^{\frac{d-1}{2}}} ds\,.$$

Proof. We use the estimate of Proposition 3.1 in conjunction with a simple convolution inequality which is based on the rapid decay of $\mathcal{F}^{-1}[\chi]$. □

4. The implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) of Theorem 1.2

Proof of (i) $\implies$ (ii). This follows from $L^{p,1}(\mu_d) \subset L^{p,\sigma}(\mu_d)$, for $\sigma \geq 1$, with continuous imbedding. $\square$

Proof of (ii) $\implies$ (iii). We use the dilation formula

$$\mathcal{B}_d[g(t^{-1}\cdot)](r) = t^d \mathcal{B}_d[g](tr). \tag{4.1}$$

If $\phi \in C_c^\infty(\mathbb{R}_+)$ then the function $f_1 := \mathcal{B}_d\phi$ belongs to $L^{p,1}(\mu_d)$ for all p and has positive norm. Now set $f_t = t^{-d(1-1/p)}\mathcal{B}_d[\phi(t^{-1}\cdot)]$; then the $L^{p,1}(\mu_d)$ norm of f_t is independent of t. Let $\|m\|$ denote the $L^{p,1}(\mu_d) \to L^{q,\sigma}(\mu_d)$ operator norm of T_m. We may estimate

$$\begin{aligned}
&\|f_1\|_{L^{p,1}(\mu_d)}\|m\| = \|f_t\|_{L^{p,1}(\mu_d)}\|m\| \geq \|\mathcal{B}_d[m\mathcal{B}_d f_t]\|_{L^{q,\sigma}(\mu_d)} \\
&= t^{-d(1-1/p)}\|\mathcal{B}_d[\phi(t^{-1}\cdot)m]\|_{L^{q,\sigma}(\mu_d)} = t^{d(1/p-1/q)}\big\|\mathcal{B}_d[m(t\cdot)\phi]\big\|_{L^{q,\sigma}(\mu_d)}
\end{aligned}$$

which proves the implication. $\square$

Proof of (iii) $\implies$ (iv). Let $u_{\mathrm{ev}}(t,\rho)$ the even extension of $\phi(\rho)m(t\rho)$ to $\mathbb{R}$. Let $h_t := \mathcal{F}_{\mathbb{R}}^{-1}[u_{\mathrm{ev}}(t,\cdot)]$. We claim that it suffices to show

$$\big\|(1+|\cdot|)^{-\frac{d-1}{2}}h_t\big\|_{L^{q,\sigma}(\nu)} \lesssim \big\|\mathcal{B}_d[\phi m(t\cdot)]\big\|_{L^{q,\sigma}(\mu_d)}, q<2, \tag{4.2}$$

where $d\nu(x) = (1+|x|)^{d-1}dx$. Indeed if (4.2) holds let $\zeta \in \mathcal{S}(\mathbb{R})$ so that $\widehat{\zeta}$ is supported in $(1/4,4)$ and $\widehat{\zeta}(\rho) = 1$ on $[1/2,2]$. Then $k_t = \zeta * h_t$ and an application of (2.11) shows that we can replace h_t by k_t in (4.2).

We proceed to show (4.2). First observe that $\phi m(t\cdot)$ is an L^2 function; namely by assumption (iii) and the Hausdorff-Young inequality it belongs to the dual space of $L^{q,\sigma}(\mu_d)$ and in view of its support to L^2. Since $\mathcal{B}_1$ is the cosine transform and since $\mathcal{B}_d^2$ is the identity on L^2 functions the inequality (4.2) follows from

$$\big\|(1+(\cdot))^{-\frac{d-1}{2}}\mathcal{B}_1[\chi\mathcal{B}_d g]\big\|_{L^{q,\sigma}((1+r)^{d-1}dr)} \lesssim \|g\|_{L^{q,\sigma}(\mu_d)}, \quad q \leq 2, \tag{4.3}$$

applied to $g = \mathcal{B}_d[\phi m(t\cdot)]$. Here the function χ is assumed to be smooth and supported in $(1/4,4)$ and equal to one on the support of ϕ. This inequality is related to and could be derived from the more sophisticated transplantation theorems of Stempak [35] and Nowak and Stempak [24] on the composition of nonmodified Hankel transforms, but (4.3) has an easy direct proof: We first note that (4.3) follows by real interpolation from the L^q inequalities, i.e. the case $q=\sigma$. Thus it suffices to show

$$\big\|\mathcal{B}_1[\chi\mathcal{B}_d g]\big\|_{L^q((1+r)^{(d-1)(1-q/2)}dr)} \lesssim \|g\|_{L^q(\mu_d)}. \tag{4.4}$$

This in turn follows easily from Corollary 3.2 and an estimate of Hardy type. Indeed changing variables $s = r+u$ and an application of Minkowski's inequality yields

$$\begin{aligned}&\big\|\mathcal{B}_1[\chi\mathcal{B}_d g]\big\|_{L^q((1+r)^{(d-1)(1-q/2)}dr)}\\ &\lesssim \int_{-\infty}^{\infty}(1+|u|)^{-N}\Big(\int_{r=-u}^{\infty}(1+r)^{(d-1)(1-q/2)}\frac{|f(r+u)|^q(r+u)^{(d-1)q}}{(1+|r+u|)^{\frac{d-1}{2}q}}dr\Big)^{1/q}du.\end{aligned}$$

We use the estimate $(1+r)^\alpha \lesssim (1+|r+u|)^\alpha(1+|u|)^\alpha$ for $\alpha = (d-1)(1-q/2)$. Thus the last displayed term is seen to be bounded by

$$C\int_{-\infty}^{\infty}(1+|u|)^{-N+(d-1)(1-q/2)}\Big(\int_{-u}^{\infty}|f(r+u)|^q(r+u)^{d-1}dr\Big)^{1/q}du$$

which for large N is $\lesssim \|f\|_{L^q(\mu_d)}$. This shows (4.4) and finishes the proof of the implication $(iii) \implies (iv)$. □

5. Sufficiency: The basic decomposition

In this section we begin the proof of the main implication (iv) $\implies$ (i) of Theorem 1.2. Let $\varphi \in C_c^\infty(\frac{1}{2}, 2)$ as in (2.14). Let $\kappa_j(r) = \mathcal{F}_{\mathbb{R}}^{-1}[\varphi m(2^j\cdot)]$, let

$$A_j(q,\sigma) = \big\|(1+|\cdot|)^{-\frac{d-1}{2}}\kappa_j\big\|_{L^{q,\sigma}(\nu)} \tag{5.1}$$

with $d\nu = (1+|x|)^{d-1}dx$, and

$$A \equiv A(p,q,\sigma) := \sup_j 2^{jd(\frac{1}{p}-\frac{1}{q})}A_j(q,\sigma). \tag{5.2}$$

Define

$$K_j = \mathcal{K}_{d,d}[\varphi m(2^j\cdot)] \tag{5.3}$$

(*cf.* (3.2)) and

$$T^j f(r) = \int 2^{jd}K_j(2^j r, 2^j s)f(s)s^{d-1}ds. \tag{5.4}$$

Define Littlewood-Paley cutoffs L_j, $\widetilde{L}_j$ by $\mathcal{B}_d[L_j f](\rho) = \varphi(2^{-j}\rho)\mathcal{B}_d f(\rho)$ and $\mathcal{B}_d[\widetilde{L}_j f](\rho) = \eta(2^{-j}\rho)\mathcal{B}_d f(\rho)$ where η is supported in $(1/4,4)$ and equal to 1 on the support of φ. Then $\mathcal{B}_d[m\mathcal{B}_d f] = \sum_j L_j T^j \widetilde{L}_j f$. We apply (the Lorentz space analogues of) the Littlewood-Paley inequalities (2.3), (2.4) (one with the L_j, the other one with the $\widetilde{L}_j$). Using also Lemma 2.4 (which justifies the use of the specific cutoff function φ in (2.14)) we see that Theorem 1.2 follows from the inequalities for vector-valued functions $\{f_j\}_{j\in\mathbb{Z}}$,

$$\Big\|\Big(\sum_j |T^j f_j|^2\Big)^{1/2}\Big\|_{L^{q,\sigma}(\mu_d)} \lesssim A(p,q,\sigma)\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^{p,\omega}(\mu_d)}. \tag{5.5}$$

For a further decomposition we introduce the notation

$$\chi_n(r) = \chi_{[2^n,2^{n+1}]}(r)$$

and decompose a.e. into three parts

$$T^j f = \sum_{n\in\mathbb{Z}} \chi_n \Big(\sum_{m<j+n-5} + \sum_{\substack{j+n-5\le m\\ \le j+n+5}} + \sum_{m>j+n+5} T^j[f\chi_{m-j}]\Big). \tag{5.6}$$

The first term will contribute to a Hardy type (or Hilbert integral type) operator whose estimate needs the full strength of the assumption. The second term will contribute to a singular integral operator, for vector-valued functions, whose estimation however will not require the full strength of our assumption. We consider the third term as an "error" term which contributes again to a better behaved Hardy type operator.

We let

$$H_{j,m} f = \sum_{n>m-j+5} \chi_n T^j[\chi_{m-j} f], \tag{5.7}$$

$$S_{j,n,i} f = \chi_n T^j[\chi_{n+i} f], \tag{5.8}$$

$$E_{j,m} f = \sum_{n<m-j-5} \chi_n T^j[\chi_{m-j} f]. \tag{5.9}$$

By (5.6)

$$T^j = \sum_{m\in\mathbb{Z}} H_{j,m} + \sum_{n\in\mathbb{Z}} \sum_{i=-5}^{5} S_{j,n,i} + \sum_{m\in\mathbb{Z}} E_{j,m}.$$

We now state the main estimates regarding these three terms. The implicit constants may depend on the parameters $p, q, \sigma, \varepsilon, d$. For the main term we have

Proposition 5.1. *For $m \in \mathbb{Z}$, $1 < p \le q < 2$, $1 \le \sigma \le \infty$*

$$\begin{aligned} &\big\|H_{j,m} f\|_{L^{q,\sigma}(\mu_d)} \\ &\qquad \lesssim \min\{2^{-m(d(\frac{1}{p}-\frac{1}{2})-\frac{1}{2})}, 2^{m\frac{d}{p'}}\} 2^{jd(\frac{1}{p}-\frac{1}{q})} A_j(q,\sigma)\|f\|_{L^{p,\infty}}(\mu_d). \end{aligned} \tag{5.10}$$

Note that in the range of interest, $1 < p < \frac{2d}{d+1}$, these estimates can be summed in m.

The estimation of the remaining two terms (5.8), (5.9) does not need the full strength of our assumptions. To formulate the appropriate weaker

hypotheses let, for $\varepsilon \ge 0$, $1 \le u < 2$

(5.11)
$$B_j(\varepsilon, u) = \Big(\int_{-\infty}^{\infty} |\kappa_j(x)|^u (1+|x|)^{u\varepsilon} dx \Big)^{1/u},$$

(5.12)
$$B(\varepsilon, p, q) = \sup_j 2^{jd(1/p-1/q)} B_j(\varepsilon, u(p,q)), \text{ where } \frac{1}{u(p,q)} = \frac{\frac1p + \frac1q - 1}{\frac2p - 1}.$$

Proposition 5.2. *Let $\varepsilon > 0$, $1 < p \le q < 2$, $1 \le \sigma \le \infty$, and let $\theta \equiv \theta(p,q) = (\frac1p - \frac1q)/(\frac1p - \frac12)$. For $m \in \mathbb{Z}$,*

$$\begin{aligned} \|E_{j,m} f\|_{L^{q,\sigma}(\mu_d)} \\ \lesssim B(4\varepsilon(1-\theta), p, q) \min\{2^{-m(1-\theta)\varepsilon}, 2^{m(1-\theta)(d-1)}\} \|f\|_{L^{p,\sigma}(\mu_d)}. \end{aligned} \tag{5.13}$$

The square-function estimates associated to $\{S_{j,n,i}\}_{j\in\mathbb{Z}}$ can be seen as estimates for vector-valued singular integrals under the assumption $B(\varepsilon, p, q) < \infty$, for small $\varepsilon > 0$.

Proposition 5.3. *For $n \in \mathbb{Z}$, $-5 \le i \le 5$, $1 < p < 2$,*

$$\Big\| \Big(\sum_j |S_{j,n,i} f_j|^2 \Big)^{1/2} \Big\|_{L^{q,\sigma}(\mu_d)} \lesssim B(\varepsilon, p, q) \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_{L^{p,\sigma}(\mu_d)} \tag{5.14}$$

To see that the conditions of Propositions 5.2 and 5.3 are less restrictive than the condition (1.8) we note

Lemma 5.4. *Suppose $p < \frac{2d}{d+1}$, $p \le q < 2$, and $\frac1u = \frac{p^{-1}+q^{-1}-1}{2p^{-1}-1}$. Then there is $\varepsilon = \varepsilon(p,q) > 0$ so that $B_j(\varepsilon, u) \lesssim A_j(q, \sigma)$, for all $\sigma \le \infty$.*

Proof. We begin by observing that $(1+|x|)^{-\alpha}$ belongs to the Lorentz space $L^{\rho,1}(\nu)$ if and only if $\alpha\rho > d$. Now write

$$B_j(\varepsilon, u) = \Big(\int \frac{|\kappa(x)|^u}{(1+|x|)^{u\frac{d-1}{2}}} (1+|x|)^{\varepsilon u + \frac{d-1}{2} u + 1 - d} d\nu(x) \Big)^{1/u}$$

with $d\nu(x) = (1+|x|)^{d-1}$. Note that by assumption the $L^{q/u,\infty}(\nu)$ norm of $|\kappa_j|^u (1+|x|)^{-u(d-1)/2}$ is bounded by $A_j(q,\infty)^u$. Thus it suffices to check that for sufficiently small ε the function

$$V_\varepsilon(x) = (1+|x|)^{\varepsilon u + \frac{d-1}{2} u + 1 - d}$$

belongs to $L^{(q/u)',1}(\nu)$. This holds under the condition $(d-1)(1-u/2) > d(1-u/q)$. Since $u^{-1} = \frac{p^{-1}+q^{-1}-1}{2p^{-1}-1}$ a straightforward computation shows that the condition is equivalent to an inequality which is independent of $q \in [p, 2)$, namely just $p < 2d/(d+1)$. $\square$

For later use let us also observe that $B(\varepsilon) \equiv B(\varepsilon, p, p)$ is independent of p, namely

$$B(\varepsilon) = \sup_j \|\kappa_j\|_{L^1((1+|x|)^\varepsilon dx)}. \tag{5.15}$$

Moreover for some real interpolations, we shall need the following locally uniform control on the constants $B(\varepsilon, p, q)$.

Lemma 5.5. *Let $1 < p \le q < 2$ and $\varepsilon > 0$. Then there exist constants $C, \eta > 0$ (depending on ε, p, q) so that for all $\tilde p \in (p - \eta, p + \eta)$ and $\tilde q$ satisfying $\frac{1}{\tilde p} - \frac{1}{\tilde q} = \frac{1}{p} - \frac{1}{q}$ we have*

$$B(\varepsilon/2, \tilde p, \tilde q) \le C\, B(\varepsilon, p, q).$$

Proof. We first observe that when $u_1 \ge u$

$$B_j(\varepsilon, u_1) \le C_\varepsilon\, B_j(\varepsilon, u). \tag{5.16}$$

Indeed, this follows from the fact that the Fourier transform of κ_j is compactly supported and therefore can be written as a convolution with a Schwartz function; we then apply Lemma 2.2.

On the other hand, if $u_1 < u$, by Hölder's inequality we have

$$B_j(\varepsilon/2, u_1) \le C_{\varepsilon, u, u_1}\, B_j(\varepsilon, u), \tag{5.17}$$

provided we choose $\frac{1}{u_1} < \frac{1}{u} + \frac{\varepsilon}{2}$. Now let $p - \eta < \tilde p < p + \eta$ and define $\tilde q$ so that $\tilde p^{-1} - \tilde q^{-1} = p^{-1} - q^{-1}$, where $\eta = \eta(\varepsilon, p, q) > 0$ is chosen so that $|u(\tilde p, \tilde q)^{-1} - u(p, q)^{-1}| < \varepsilon/4$. Then using either (5.16) or (5.17) and $\tilde p^{-1} - \tilde q^{-1} = p^{-1} - q^{-1}$ the asserted estimate follows. $\square$

Proof of Theorem 1.2, given Propositions 5.1, 5.2, 5.3. We need to estimate the square-function on the left hand side of (5.5) with T^j replaced by one of the terms $\sum_m H_{j,m}$, $\sum_{n\in\mathbb{Z}} \sum_{i=-5}^{5} S_{j,n,i}$, and $\sum_m E_{j,m}$.

Observe that $H_{j,m} f_j = H_{j,m}[f_j \chi_{m-j}]$ and we bound

$$\Big\| \Big(\sum_j \Big| \sum_m H_{j,m} f_j \Big|^2 \Big)^{1/2} \Big\|_{L^{q,\sigma}(\mu_d)} \le \sum_m \Big\| \Big(\sum_j |H_{j,m}[f_j \chi_{m-j}]|^2 \Big)^{1/2} \Big\|_{L^{q,\sigma}(\mu_d)}$$
$$\le \sum_m \Big(\sum_j \|H_{j,m}[f_j \chi_{m-j}]\|^{\omega}_{L^{q,\sigma}(\mu_d)} \Big)^{1/\omega}, \quad \omega = \min\{q, \sigma\}.$$

Here we have used Minkowski's inequality for the m-summation, followed by Lemma 2.1. Let $\delta(p) = \min\{d/p', d(1/p - 1/2) - 1/2\}$ then $\delta(p) > 0$ for $1 < p < \frac{2d}{d+1}$ and by Proposition 5.1 the last expression in the displayed

formula is bounded by $CA(p,q,\sigma)$ times

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} 2^{-|m|\delta(p)}\Big(\sum_j \big\|f_j\chi_{m-j}\big\|^{\omega}_{L^{p,\infty}(\mu_d)}\Big)^{1/\omega}\\
&\lesssim \sum_{m\in\mathbb{Z}} 2^{-|m|\delta(p)}\Big(\sum_j \big\|f_j\chi_{m-j}\big\|^{\omega}_{L^{p,\omega}(\mu_d)}\Big)^{1/\omega}\\
&\lesssim \sum_{m\in\mathbb{Z}} 2^{-|m|\delta(p)}\Big\|\sup_j |f_j\chi_{m-j}|\Big\|_{L^{p,\omega}(\mu_d)}\\
&\lesssim \Big\|\sup_j |f_j|\Big\|_{L^{p,\omega}(\mu_d)} \lesssim \Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^{p,\omega}(\mu_d)}.
\end{aligned}$$

Here, in order to bound the second expression, we have used (2.5), and the assumption that $\omega \ge p$, together with the disjointness of the intervals $[2^{m-j}, 2^{m-j+1})$. This completes the proof of the $L^{p,\omega}(\ell^2,\mu_d) \to L^{q,\sigma}(\ell^2,\mu_d)$ bound for $\{\sum_m H_{j,m}f_j\}_{j\in\mathbb{Z}}$. The terms $\{\sum_m E_{j,m}f_j\}_{j\in\mathbb{Z}}$ are estimated similarly, given Proposition 5.2 and Lemma 5.4.

Concerning the terms $S_{j,n,i}$, let us consider the $L^p \to L^q$ estimates. We recall $S_{j,n,i}f_j = \chi_n S_{j,n,i}[f_j\chi_{n+i}]$ and use Proposition 5.3, for fixed i, and n. In view of the cutoffs $\chi_n(r)$, $\chi_{n+i}(s)$, $-5 \le i \le 5$ the uniform Lebesgue space estimate of Proposition 5.3 also gives an $L^p(\mu_d)$ estimate for the sum,

$$\Big\|\Big(\sum_j\Big|\sum_n S_{j,n,i}f_j\Big|^2\Big)^{1/2}\Big\|_{L^q(\mu_d)} \le C_{\varepsilon,p}B(\varepsilon,p,q)\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^p(\mu_d)}.$$

We sum in $i \in \{-5,\dots,5\}$ and by Lemma 5.4 we obtain the desired $L^p \to L^q$ estimate for the singular integral part in the range $1 < p < 2d/(d+1)$. By real interpolation (and Lemma 5.5) this extends to the $L^{p,\sigma} \to L^{q,\sigma}$ estimates.

6. Proof of Proposition 5.1

Let $I_n = [2^n, 2^{n+1}]$, and $\mathcal{R}_n = [2^n, \infty)$. We estimate

$$\begin{aligned}
&\Big\|\sum_{n>m-j+5}\chi_n T^j[f\chi_{m-j}]\Big\|_{L^{q,\sigma}(\mu_d)}\\
&\le \Big\|\chi_{\mathcal{R}_{m-j+5}}\int 2^{jd}|\mathcal{K}_j(2^j\cdot, 2^js)||f(s)|\chi_{m-j}(s)s^{d-1}ds\Big\|_{L^{q,\sigma}(\mu_d)}\\
&= 2^{-jd/q}\Big\|\chi_{\mathcal{R}_{m+5}}\int_{I_m}|\mathcal{K}_j(\cdot,s)||f(2^{-j}s)|s^{d-1}ds\Big\|_{L^{q,\sigma}(\mu_d)} \qquad (6.1)
\end{aligned}$$

by changes of variables in s and r.

We now use the kernel estimate of Proposition 3.1 and set

$$W_j(x) = \int \frac{|\kappa_j(x-u)|}{(1+|u|)^N} du. \tag{6.2}$$

We apply Minkowski's inequality (i.e. the continuous form of the triangle inequality in the Lorentz space $L^{q,\sigma}$ which is a Banach space) and see that the expression (6.1) is controlled by

$$2^{-jd/q} \int_{I_m} |f(2^{-j}s)| \frac{s^{d-1}}{(1+s)^{\frac{d-1}{2}}} \sum_{(\pm,\pm)} \Big\| \chi_{\mathcal{R}_{m+5}} \frac{W_j(\pm\cdot\pm s)}{(1+\cdot)^{\frac{d-1}{2}}} \Big\|_{L^{q,\sigma}(\mu_d)} ds.$$

It is now crucial that in the inner norm the functions are restricted to the set where $r \ge 2^{m+5}$ while $s \le 2^{m+1}$. We may therefore change variables and use the bound $(1+|r-s|) \ge c(1+r)$ in this range, so that

$$\Big\| \chi_{\mathcal{R}_{m+5}} \frac{W_j(\pm\cdot\pm s)}{(1+\cdot)^{\frac{d-1}{2}}} \Big\|_{L^{q,\sigma}(\mu_d)} \lesssim \Big\| \frac{W_j}{(1+|\cdot|)^{\frac{d-1}{2}}} \Big\|_{L^{q,\sigma}(\nu)}, \quad s \le 2^{m+1},$$

where $d\nu = (1+|x|)^{d-1}dx$. By Lemma 2.3 the term on the right hand side is also controlled by $\big\|\kappa_j(1+|\cdot|)^{-\frac{d-1}{2}}\big\|_{L^{q,\sigma}(\nu)}$, which is $A_j(q,\sigma)$.

Thus we see that the expression (6.1) is bounded by

$$C2^{jd(1/p-1/q)} A_j(q,\sigma) \int_{I_m} 2^{-jd/p} |f(2^{-j}s)|(1+s)^{-(d-1)/2} s^{d-1} ds.$$

It remains to bound the s-integral. It is easy to check that the restriction of $\Omega(s) = (1+s)^{-(d-1)/2}$ to the interval I_m belongs to $L^{p',1}(I_m,\mu_d)$ and satisfies the bounds

$$\big\|\chi_m \Omega\big\|_{L^{p',1}(\mu_d)} \lesssim \begin{cases} 2^{-m(d(1/p-1/2)-1/2)} & \text{if } m \ge 0, \\ 2^{md/p'} & \text{if } m \le 0, \end{cases}$$

and thus, by duality

$$\int_{I_m} 2^{-jd/p} |f(2^{-j}s)| \frac{s^{d-1}}{(1+s)^{\frac{d-1}{2}}} ds \le \|\chi_m\Omega\|_{L^{p',1}(\mu_d)} \|2^{-jd/p} f(2^{-j}\cdot)\|_{L^{p,\infty}(\mu_d)}$$
$$\lesssim \min\{2^{-m(d(1/p-1/2)-1/2)}, 2^{md/p'}\} \|f\|_{L^{p,\infty}(\mu_d)}.$$

This finishes the proof. □

7. More L^p estimates

In this section we consider the case $p = q$ of Propositions 5.2 and 5.3; the general case will be handled in §9. The results of this section together with the previous section complete the proof of Theorem 1.2 in the case $p = q$. In what follows we shall assume $p = \sigma$ in the proof of Proposition 5.3 since the $L^{p,\sigma}$ boundedness results follow then by interpolation and replacing ε

with $\varepsilon/2$. Moreover we prove the statement of Proposition 5.2 for the case $p = q = \sigma$ with the constant with the constant $B(\varepsilon)$ (rather than $B(4\varepsilon)$), and the factor 4 is included in the statement of the proposition to account for interpolations needed for the general case (*cf.* also Lemma 5.5).

Proof of Proposition 5.2, $p = q = \sigma$**.** We begin with the estimate (6.1) which is still valid but continue differently since now $n + j \le m - 5$, thus $r \ll s$. Let $I_m^* = [2^{m-1}, 2^{m+2}]$. Set $h_{p,j}(s) = 2^{-jd/p} f(2^{-j}s)s^{\frac{d-1}{p}}$. Then the right hand side of (6.1) is estimated by

$$\sum_{(\pm,\pm)}\Big(\sum_{n\le m-j-5}\int_{I_{n+j}}\Big|\int_{I_m}\frac{|W_j(\pm r\pm s)|}{(1+s)^{\frac{d-1}{2}}}\frac{h_{p,j}(s)s^{(d-1)/p'}}{(1+r)^{\frac{d-1}{2}}}ds\Big|^p r^{d-1}dr\Big)^{1/p}$$

$$\lesssim \sum_{(\pm,\pm)}\Big(\int_0^{2^{m-3}}\Big|\int_{I_m^*}\frac{|W_j(\pm y)|}{(1+y)^{\frac{d-1}{2}}}[\chi_m h_{p,j}](y\pm r)dy\Big|^p\frac{2^{m(d-1)\frac{p}{p'}}r^{d-1}}{(1+r)^{\frac{d-1}{2}p}}dr\Big)^{1/p}.$$

If $m > 0$ this is dominated by

$$C\sum_{\pm}2^{-m\varepsilon}\int|W_j(y)|(1+|y|)^{\varepsilon}\Big(\int\big|[\chi_m h_{p,j}](y\pm r)\big|^p dr\Big)^{1/p}dy$$

$$\lesssim 2^{-m\varepsilon}\|\kappa_j\|_{L^1((1+|\cdot|)^\varepsilon dy)}\|f\chi_{m-j}\|_{L^p(\mu_d)}.$$

If $m < 0$ we may instead estimate $2^{m(d-1)p/p'}r^{d-1} \le 2^{m(d-1)p}$; this yields the bound

$$2^{m(d-1)}\|\kappa_j\|_1\|f\chi_{m-j}\|_{L^p(\mu_d)}$$

instead. This finishes the proof. □

Proof of Proposition 5.3, $p = q = \sigma$**.** We use standard arguments for singular integrals for ℓ^2-valued kernels and functions. First, by orthogonality,

$$\Big\|\Big(\sum_j|S_{j,n,i}f_j|^2\Big)^{1/2}\Big\|_{L^2(\mu_d)} \le \Big\|\Big(\sum_j\big|T^j[f_j\chi_{n-i}]\big|^2\Big)^{1/2}\Big\|_{L^2(\mu_d)}$$

$$\lesssim \sup\|\widehat{\kappa}_j\|_\infty\Big\|\Big(\sum_j|f_j|^2\Big)^{1/2}\Big\|_{L^2(\mu_d)}.$$

To prove the $L^p(\mu_d)$ bounds for $1 < p < 2$ it suffices, by the Marcinkiewicz interpolation theorem, to prove the weak type $(1,1)$ inequality

$$\mu_d\Big(\Big\{r : \Big(\sum_j|S_{j,n,i}f_j|^2\Big)^{1/2} > \lambda\Big\}\Big) \lesssim B\lambda^{-1}\Big\|\Big(\sum_j|f_j|^2\Big)^{1/2}\Big\|_{L^1(\mu_d)}; \tag{7.1}$$

here $B = B(\varepsilon)$ as in (5.15).

Set $h_j(s) = f_j(s)(2^{-n}s)^{d-1}\chi_{n+i}(s)$, so that $|h_j|$ and $|f_j|$ are of comparable size on I_{n+i}. For fixed $\lambda > 0$ we make a Calderón-Zygmund decomposition of the ℓ^2 valued function $\{h_j\}$, at height λ/B (see [31]). We thus decompose

$h_j = g_j + b_j$ where $\|\vec{g}\|_{L^\infty(\ell^2)} \le \lambda/B$, $\|\vec{g}\|_{L^1(\ell^2,ds)} + \|\vec{b}\|_{L^1(\ell^2,ds)} \lesssim \|\vec{h}\|_{L^1(\ell^2,ds)}$. Furthermore $b_j = \sum_\nu b_{j,\nu}$ so that $b_{j,\nu}$ is supported in a dyadic subinterval J_ν of I_{n+i}, with center s_ν and length 2^{L_ν}. The interiors of the intervals J_ν are disjoint, and we have $|J_\nu|^{-1}\int_{J_\nu} |\vec{b}_\nu(s)|_{\ell^2} ds \lesssim \lambda/B$ and $\sum_\nu |J_\nu| \le B\lambda^{-1}\|\vec{h}\|_{L^1(\ell^2,ds)}$. Finally $\int b_{j,\nu} ds = 0$ for all j,ν.

Note that $S_{j,n,i}f_j = S_{j,n,i}\vec{\mathfrak{g}}_j + \sum_\nu S_{j,n,i}\vec{\mathfrak{b}}_{j,\nu}$ where $\mathfrak{g}_j(s) = g_j(s)(2^n/s)^{d-1}$ and $\mathfrak{b}_{j,\nu}(s) = b_{j,\nu}(s)(2^n/s)^{d-1}$. We estimate

$$\mu_d(\{r \in I_{n+i} : |\{S_{j,n,i}\mathfrak{g}_j(r)\}|_{\ell^2} > \lambda/2\}) \lesssim \lambda^{-2}B^2\|\vec{\mathfrak{g}}\|^2_{L^2(\ell^2,\mu_d)}$$
$$\lesssim \lambda^{-1}B\|\vec{\mathfrak{g}}\|_{L^1(\ell^2,\mu_d)} \lesssim \lambda^{-1}B\|\vec{f}\|_{L^1(\ell^2,\mu_d)}. \tag{7.2}$$

For each interval J_ν let J_ν^* denote the interval with same center and tenfold length. Also let $\Omega = \cup_\nu J_\nu^*$ then
(7.3)
$$\mu_d(\Omega) \lesssim 2^{n(d-1)} \sum |J_\nu| \lesssim B\lambda^{-1}2^{n(d-1)}\|\vec{h}\|_{L^1(\ell^2,ds)} \lesssim B\lambda^{-1}\|\vec{h}\|_{L^1(\ell^2,\mu_d)}.$$

It remains to estimate

$$\mu_d\Big(\Big\{r \in I_n \setminus \Omega : \Big(\sum_j \big|S_{j,n,i}\big[\sum_\nu \mathfrak{b}_{j,\nu}\big]\big|^2\Big)^{1/2} > \lambda/2\Big\}\Big) \tag{7.4}$$
$$\lesssim \lambda^{-1}\int_{I_n\setminus\Omega}\Big(\sum_j \Big|S_{j,n,i}\big[\sum_\nu \mathfrak{b}_{j,\nu}\big]\Big|^2\Big)^{1/2} r^{d-1}dr$$
$$\lesssim \lambda^{-1}2^{n(d-1)}\sum_\nu\sum_j \int_{I_n\setminus J_\nu^*} \big|S_{j,n,i}\mathfrak{b}_{j,\nu}(r)\big|dr. \tag{7.5}$$

Note that

$$S_{j,n,i}\mathfrak{b}_{j,\nu}(r) = 2^{n(d-1)}\int 2^{jd}\mathcal{K}_j(2^j r, 2^j s)b_{j,\nu}(s)ds.$$

To estimate the integral in (7.5) we distinguish the cases $j \ge -L_\nu$, $j \le -L_\nu$. Note that $L_\nu \le n+5$ as $J_\nu \subset I_{n+i}$.

If $j \ge -L_\nu$ $(\ge -n-5)$ we use the kernel estimate of Proposition 3.1 and obtain, with the notation W_j in (6.2) and $r, s \approx 2^n$

$$|S_{j,n,i}\mathfrak{b}_{j,\nu}(r)| \lesssim \sum_{\pm,\pm} \frac{2^{jd}2^{n(d-1)}}{(1+2^j r)^{\frac{d-1}{2}}(1+2^j s)^{\frac{d-1}{2}}}\int W_j(\pm 2^j r \pm 2^j s)|b_{j,\nu}(s)|ds$$
$$\lesssim \sum_{\pm,\pm}\int 2^j W_j(\pm 2^j r \pm 2^j s)|b_{j,\nu}(s)|ds$$

and if $r \notin J_\nu^*$ then $|r-s| \approx |r-s_\nu| > 2^{L_\nu}$. Consequently

$$\begin{aligned}\int_{I_n\setminus J_\nu^*} \big|S_{j,n,i}\mathfrak{b}_{j,\nu}(r)\big|dr &\lesssim \int_{|x|>2^{j+L_\nu}} |W_j|(x)dx \int |b_{j,\nu}(s)|ds \\ &\lesssim 2^{-(j+L_\nu)\varepsilon} B(\varepsilon)\big\|b_{j,\nu}\big\|_{L^1(ds)}. \end{aligned} \tag{7.6}$$

If $j < -L_\nu$ we use the cancellation of the $b_{j,\nu}$ to write

$$\begin{aligned}\big|S_{j,n,i}\mathfrak{b}_{j,\nu}(r)\big| &= 2^{n(d-1)}\Big|\int 2^{jd}\big[\mathcal{K}_j(2^jr,2^js)-\mathcal{K}_j(2^jr,2^js_\nu)\big]b_{j,\nu}(s)ds\Big| \\ &\lesssim 2^{n(d-1)}2^{j+L_\nu}\int_{\sigma=0}^1\int 2^{jd}\big|\partial_s\mathcal{K}_j(2^jr,2^j(s_\nu+\sigma(s-s_\nu)))\big|\,\big|b_{j,\nu}(s)\big|ds.\end{aligned}$$

We now argue as before, but use Proposition 3.1 to estimate $\partial_s\mathcal{K}_j$ and we obtain for $j\le -L_\nu$

$$\begin{aligned}&\int_{I_n\setminus J_\nu^*}\big|S_{j,n,i}\mathfrak{b}_{j,\nu}(r)\big|dr \\ &\lesssim \int |b_{j,\nu}(s)|ds\, 2^{j+L_\nu}2^{n(d-1)}\sum_{(\pm,\pm)}\sup_a\int\frac{2^{jd}W_j(\pm 2^jr\pm 2^ja)}{1+2^{(j+n)(d-1)}}dr \\ &\lesssim \int |b_{j,\nu}(s)|ds\, 2^{j+L_\nu}\sum_{(\pm,\pm)}\sup_a\int 2^jW_j(\pm 2^jr\pm 2^ja)dr \\ &\lesssim B(0)2^{j+L_\nu}\|b_{j,\nu}\|_{L^1(ds)}.\end{aligned} \tag{7.7}$$

We can sum the terms (7.6) and (7.7) in j and obtain

$$\begin{aligned}&\sum_j\int_{I_n\setminus J_\nu^*}\big|S_{j,n,i}\mathfrak{b}_{j,\nu}(r)\big|dr \\ &\lesssim B(\varepsilon)\sum_j \min\{2^{j+L_\nu},2^{-(j+L_\nu)\varepsilon}\}\big\|b_{j,\nu}\big\|_{L^1(ds)} \lesssim B(\varepsilon)\big\|\vec{b}_\nu\big\|_{L^1(\ell^2,ds)}.\end{aligned}$$

Now we sum in ν and get the required $L^1(\mu_d)$ bound off Ω. The expression (7.5) is thus dominated by

$$\begin{aligned}&\lambda^{-1}B(\varepsilon)\sum_\nu 2^{n(d-1)}\big\|\vec{b}_\nu\big\|_{L^1(\ell^2,ds)} \\ &\lesssim \sum_\nu |J_\nu|2^{n(d-1)} \lesssim \lambda^{-1}B(\varepsilon)2^{n(d-1)}\int_{I_n}|\vec{h}(s)|_{\ell^2}ds \\ &\lesssim \lambda^{-1}B(\varepsilon)\int_{I_n}|\vec{h}(s)|_{\ell^2}s^{d-1}ds.\end{aligned}$$

This bounds the expression (7.4) by $CB(\varepsilon)\lambda^{-1}\|\vec{f}\|_{L^1(\ell^2,\mu_d)}$. Combining this bound with (7.2) and (7.3) yields the desired weak type $(1,1)$ bound (7.1). $\square$

8. $L^p \to L^2$ ESTIMATES

In this section we prove some sharp $L^p \to L^2$ bounds for Hankel multipliers.

Theorem 8.1. *Let $d > 1$.*

(i) Suppose $1 < p < \frac{2d}{d+1}$. Then $m \in \mathfrak{M}_d^{p,2}$ if and only if

$$\sup_{t>0} t^{d(\frac{1}{p}-\frac{1}{2})} \Big(\int_t^{2t} |m(\rho)|^2 \frac{d\rho}{\rho} \Big)^{1/2} < \infty. \tag{8.1}$$

(ii) Let $p_d = \frac{2d}{d+1}$. Then the operator $T : f \mapsto \mathcal{B}_d[m\mathcal{B}_d f]$ maps the Lorentz space $L^{p_d,1}(\mu_d)$ to $L^2(\mu_d)$ if and only if (8.1) holds for $p = p_d$.

Remark. It is easy to see that the condition (8.1) is equivalent to

$$\sup_{t>0} t^{d(\frac{1}{p}-\frac{1}{2})} \|\phi m(t\cdot)\|_2 < \infty \tag{8.2}$$

for some nontrivial, smooth ϕ with compact support in $(0, \infty)$.

Proof of Theorem 8.1. We first prove (i). The necessity of the condition has already been established in §4. For the proof of the sufficiency let T^j be as in (5.4). We then show the estimate

$$\big\|T^j f\big\|_{L^2(\mu_d)} \lesssim A_j(p, 2)\|f\|_{L^p(\mu_d)} \tag{8.3}$$

where $A_j(p, 2) = 2^{jd(\frac{1}{p}-\frac{1}{2})}\|\varphi m(2^j\cdot)\|_2$. Note that by Plancherel's theorem and the argument of Lemma 2.4 the condition $\sup_j A_j(p, 2) < \infty$ is equivalent with (8.1) (and also with (8.2)). Now,

$$\begin{aligned}
&\big\|T^j f\big\|_{L^2(\mu_d)} = \Big(\int \Big[\int 2^{jd}\mathcal{K}_j(2^j r, 2^j s) f(s) s^{d-1} ds \Big]^2 r^{d-1} dr \Big)^{1/2} \\
&= 2^{-jd/2}\Big(\int \Big[\int \mathcal{K}_j(r, s) f(2^{-j}s) s^{d-1} ds \Big]^2 r^{d-1} dr \Big)^{1/2} \\
&\lesssim 2^{-jd/2} \sum_{(\pm,\pm)} \int_0^\infty |f(2^{-j}s)| \frac{s^{d-1}}{(1+s)^{\frac{d-1}{2}}} \Big(\int_0^\infty \Big| \frac{W_j(\pm r \pm s)}{(1+r)^{\frac{d-1}{2}}} \Big|^2 r^{d-1} dr \Big)^{1/2} ds
\end{aligned}$$

where for the last bound we used Minkowski's inequality and the kernel estimate from Proposition 3.1. The last expression is controlled by

$$\begin{aligned}
&2^{-jd/2}\|\kappa_j\|_2 \int |f(2^{-j}s)| \frac{s^{d-1}}{(1+s)^{\frac{d-1}{2}}} ds \\
&\qquad \le 2^{-jd/2}\|\kappa_j\|_2 \Big(\int |f(2^{-j}s)|^p s^{d-1} ds \Big)^{1/p} \Big(\int \frac{s^{d-1}}{(1+s)^{\frac{d-1}{2}p'}} ds \Big)^{1/p'}
\end{aligned} \tag{8.4}$$

and the second integral in the last line is finite for $p < \frac{2d}{d+1}$. Changing variables we obtain

$$\big\|T^j f\big\|_{L^2(\mu_d)} \lesssim 2^{jd(1/p-1/2)} \|\kappa_j\|_2 \|f\|_{L^p(\mu_d)}.$$

We now use orthogonality and Littlewood-Paley theory, writing $L_j f = \mathcal{B}_d[\chi(2^{-j}\cdot)\mathcal{B}_d f]$ and $T^j = L_j T^j L_j$ to get

$$\begin{aligned}\|Tf\|_{L^2(\mu_d)} &\lesssim \Big(\sum_j \|T^j L_j f\|^2_{L^2(\mu_d)}\Big)^{1/2} \\ &\lesssim \sup_j 2^{jd(1/p-1/2)} \|\kappa_j\|_2 \Big(\sum_k \|L_k f\|^2_{L^p(\mu_d)}\Big)^{1/2}\end{aligned}$$

and the argument is concluded by observing that for $1 < p \le 2$

$$\Big(\sum_k \|L_k f\|^2_{L^p(\mu_d)}\Big)^{1/2} \le \Big\|\Big(\sum_k |L_k f|^2\Big)^{1/2}\Big\|_{L^p(\mu_d)} \le C_p \|f\|_{L^p(\mu_d)}. \tag{8.5}$$

The proof of (ii) is largely analogous. We may assume that f is the characteristic function of a measurable set E. The difference is the estimate (8.4). We now observe that the function $\omega_d(s) = (1+s)^{-\frac{d-1}{2}}$ belongs to the space $L^{p_d',\infty}(\mu_d)$ and by the duality between $L^{p_d,1}$ and $L^{p_d',\infty}$ we use instead

$$\int |\chi_E(2^{-j}s)| \frac{s^{d-1}}{(1+s)^{\frac{d-1}{2}}} ds \lesssim \big\|\chi_E(2^{-j}\cdot)\big\|_{L^{p_d,1}(\mu_d)} \big\|\omega_d\big\|_{L^{p_d',\infty}(\mu_d)}$$

which is $\lesssim [2^{jd}\mu_d(E)]^{1/p}$. The subsequent Littlewood-Paley argument is the same; we use $f = \chi_E$ in (8.5). □

Sharpness. The restricted strong type $(p_d, 2)$-estimate is sharp, as the Lorentz space $L^{p_d,1}$ cannot be replaced by $L^{p_d,\sigma}$ for $\sigma > 1$. To see this let $m_N(\rho) = \sqrt{N}\chi_{[1,1+cN^{-1}]}$ so that the condition (8.1) is satisfied uniformly in N. Let $f_N(s) = s^{-(d+1)/2} e^{-is} \chi_{[1,N]}(s)$. Then one computes that

$$\|f_N\|_{L^{p_d,\sigma}(\mu_d)} \lesssim (\log N)^{1/\sigma}$$

and using the asymptotic expansion (2.1) one computes that

$$\mathcal{B}_d f_N(\rho) = c \int_1^N e^{i(\rho-1)s} \frac{ds}{s} + O(1)$$

for ρ near 1 (observe that the corresponding integral with phase $-(\rho+1)s$ is bounded near $\rho = 1$, by an integration by parts). Thus $|\mathcal{B}_d f_N(\rho)| \gtrsim \log N$ for $|\rho - 1| \le cN^{-1}$ (if c is sufficiently small). Consequently

$$\|\mathcal{B}_d[m_N \mathcal{B}_d f_N]\|_{L^2(\mu_d)} \approx \|m_N \mathcal{B}_d f_N\|_2 \gtrsim \log N$$

which implies the assertion.

Analogue for radial Fourier multipliers. We also note that an analogue of Theorem (8.1) holds for radial Fourier multipliers acting on general $L^p(\mathbb{R}^d)$ functions, namely there is the 'folk' result

Observation 8.2. *Suppose that $1 < p \le \frac{2(d+1)}{d+3}$. Then the operator $f \mapsto \mathcal{F}^{-1}[m(|\cdot|)\widehat{f}]$ extends to a bounded operator from $L^p(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ if and only if* (8.1) *holds.*

Proof. The necessity has been observed in §4. If m_t is supported in $\{\xi : t \le |\xi| \le 2t\}$ then it follows by a well known argument of Fefferman [11] from the Stein-Tomas restriction theorem ([32], ch.IX-2) that

$$\begin{aligned}\|\mathcal{F}^{-1}[m_t(|\cdot|)\widehat{f}]\|_2 &\lesssim \Big(\int_t^{2t} |m_t(r)|^2 \int_{S^{d-1}} |\widehat{f}(r\xi')|^2 d\sigma(\xi') r^{d-1} dr\Big)^{1/2} \\ \lesssim \Big(\int_t^{2t} |m_t(r)|^2 \|\tfrac{1}{r^d} f(\tfrac{\cdot}{r})\|_p^2 r^{d-1} dr\Big)^{\frac12} &= \|f\|_p \Big(\int_t^{2t} |m_t(r)|^2 r^{2(\frac{d}{p}-\frac{d}{2})} \frac{dr}{r}\Big)^{\frac12}.\end{aligned}$$

For global multipliers the result follows now by Littlewood-Paley theory exactly as in the proof of Theorem 8.1. □

We note that the restriction $p \le \frac{2(d+1)}{d+3}$ for the result on general L^p functions is optimal as follows from the usual Knapp counterexamples for the restriction theorem.

9. Conclusion of the proof

In order to finish the proof of Theorem 1.2 it just remains to establish the $L^p(\mu_d) \to L^q(\mu_d)$ estimates in Propositions 5.2 and 5.3 for $p < q < 2$. The appropriate $L^{p,\sigma}(\mu_d) \to L^{q,\sigma}(\mu_d)$ follow then by the real interpolation method, if we take into account Lemma 5.5.

The interpolations follow results on bilinear interpolation with the complex methods (i.e. in disguise versions of Stein's interpolation theorem for analytic families), see Theorems 4.4.1 and 4.4.2 in [2]. Using the first (and more elementary) of these results we interpolate the inequalities

$$\begin{aligned}&\big\|E_{j,m} f\big\|_{L^p(\mu_d)} \lesssim \min\{2^{-m\varepsilon}, 2^{m(d-1)}\} \|\kappa_j\|_{L^1((1+|x|)^\varepsilon dx)} \|f\|_{L^p(\mu_d)}, \\ &\big\|E_{j,m} f\big\|_{L^2(\mu_d)} \lesssim 2^{jd(1/p-1/2)} \|\kappa_j\|_{L^2} \|f\|_{L^p(\mu_d)},\end{aligned}$$

where the first bound has been already been established in §7 and the second is immediate from (8.3). Similarly for the singular integrals we interpolate

$$\Big\|\Big(\sum_j |S_{j,n,i}f_j|^2\Big)^{1/2}\Big\|_{L^p(\mu_d)} \lesssim \sup_j \|\kappa_j\|_{L^1((1+|x|)^\varepsilon dx)}\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^p(\mu_d)},$$
$$\Big\|\Big(\sum_j |S_{j,n,i}f_j|^2\Big)^{1/2}\Big\|_{L^2(\mu_d)} \lesssim \sup_j 2^{jd(\frac1p-\frac12)}\|\kappa_j\|_2\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^p(\mu_d)},$$

where again the first inequality has been proved in §7 and the second follows from (8.3) and Minkowski's inequality. In order to obtain the interpolated $L^p(\mu_d) \to L^q(\mu_d)$ statements we use Lemma 2.5, and Theorem 4.4.2 in [2] (which involves the $[\cdot,\cdot]^\vartheta$ functor on one of the entries). The proof is complete. $\square$

We remark that for the interpolation of the singular operators one could have also based the proof on the more elementary Theorem 4.4.1 in [2] which only involves the $[\cdot,\cdot]_\vartheta$ method; one then has to use the fact that the space of $L^p(\mu_d)$ functions f for which $\mathcal{B}_d f$ has compact support in $(0,\infty)$ is dense in $L^p(\mu_d)$, see [36]. Thus one can reduce matters to uniform estimates for compactly supported multipliers and apply the interpolation result on the spaces $LF_o(p,a,b)$ mentioned in the remark following Lemma 2.5.

10. Miscellanea

Proof of Corollary 1.5. The $L^q((1+|r|)^{(d-1)(1-q/2)}dr)$ norm of a function κ is dominated using Hölder's inequality by

$$\Big(\sum_{j=0}^\infty \|\kappa\|^q_{L^q(\mathcal{I}_j)} 2^{j(d-1)(\frac1q-\frac12)q}\Big)^{1/q} \lesssim \Big(\sum_{j=0}^\infty \|\kappa\|^q_{L^2(\mathcal{I}_j)} 2^{jd(\frac1q-\frac12)q}\Big)^{1/q}.$$

This is applied to $\kappa = \mathcal{F}^{-1}[\phi m(t\cdot)]$ and the result follows from the definition of the Besov space. $\square$

Proof of Corollary 1.4. This is an immediate consequence of theorem 1.2 and the interpolation formula of Lemma 2.5, with varying a, b (we set $a_i = (d_i-1)(1/q_i-1/2)$ and $b_i = d(1/p_i-1/q_i)$ for $i=0,1$). $\square$

Real interpolation. We can also prove some interpolation results using the real method, in view of the nature of our conditions these are limited to the $K_{\vartheta,\infty}$ method with a number of restrictions (see [2] for general references about real interpolation).

Define $\mathfrak{M}_d^{p,q,\sigma}$ as the space of all locally integrable functions m on $\mathbb{R}_+$ for which T_m extends to a bounded operator from $L^p(\mu_d)$ to $L^{q,\sigma}(\mu_d)$; the norm is given by the operator norm of T_m. Thus $\mathfrak{M}_d^{p,q} = \mathfrak{M}_d^{p,q,q}$.

Theorem 1.1 is used to prove that for fixed d the weak type multiplier spaces $\mathfrak{M}_{d,\infty}^{p,p}$, $1 < p < 2d/(d+1)$, are stable under real interpolation, with respect to the $K_{\vartheta,\infty}$ method.

Corollary 10.1. *Suppose* $1 < d < \infty$, $1 < p_i < \frac{2d}{d+1}$, $p_i \le q_i \le 2$, *for* $i = 0,1$, *moreover* $p_0 \ne p_1$, $p_0^{-1} - q_0^{-1} = p_1^{-1} - q_1^{-1}$. *Then*

$$[\mathfrak{M}_d^{p_0,q_0,\sigma_0}, \mathfrak{M}_d^{p_1,q_1,\sigma_1}]_{\vartheta,\infty} = \mathfrak{M}_d^{p,q,\infty}, \tag{10.1}$$

for $(1/p, 1/q) = (1-\vartheta)(1/p_0, 1/q_0) + \vartheta(1/p_1, 1/q_1)$ *with* $0 < \vartheta < 1$.

Proof of Corollary 10.1. We first observe that for a compatible pair of Banach spaces A_0, A_1 we have the formula

$$[\ell_b^\infty(A_0), \ell_b^\infty(A_1)]_{\vartheta,\infty} = \ell_b^\infty([A_0, A_1]_{\vartheta,\infty}) \tag{10.2}$$

This follows quickly from the definition of the $K_{\vartheta,\infty}$ method (and interchanging two suprema).

We now set $w(r) = (1+|r|)^{-(d-1)/2}$, $d\nu(r) = (1+|r|)^{d-1}$, and let $L^{q,\sigma}(w, d\nu)$ be the space of functions f for which fw belongs to Lorentz space $L^{q,\sigma}(d\nu)$ (and the norm is given by $\|fw\|_{L^{q,\sigma}(d\nu)}$ where we work with a suitable norm on the Lorentz space). The standard interpolation formulas for Lorentz spaces apply and by (10.2) we have for $q_0 \ne q_1$ and $1/q = (1-\vartheta)/q_0 + \vartheta/q_1$,

$$[\ell_b^\infty(L^{q_0,\sigma_0}(w,d\nu)), \ell_b^\infty(L^{q_1,\sigma_1}(w,d\nu))]_{\vartheta,\infty} = \ell_b^\infty(L^{q,\infty}(w,d\nu)).$$

Now let $\mathrm{LF}_b^{q,\sigma}(w,d\nu)$ be the space of all m which are integrable over every compact subinterval of $(0,\infty)$ and satisfy the condition

$$\sup_{t>0} t^b \big\| \mathcal{F}_{\mathbb{R}}^{-1}[\phi m(t\cdot)] \big\|_{L^{q,\sigma}(w,d\nu)} < \infty.$$

Then the arguments in the proof of Lemma 2.5 show that the maps $\mathfrak{A}$, $\mathfrak{B}$ defined in (2.19), (2.20) can be used to show that $\mathrm{LF}_b^{q,\sigma}(w,d\nu)$ is a retract of $\ell_b^\infty(L^{q,\sigma}(w,d\nu))$. One deduces quickly that for $q_0 \ne q_1$

$$[\mathrm{LF}_b^{q_0,\sigma_0}(w,d\nu), \mathrm{LF}_b^{q_1,\sigma_1}(w,d\nu)]_{\vartheta,\infty} = \mathrm{LF}_b^{q,\infty}(w,d\nu)$$

and the asserted result follows from Theorem 1.2 if we apply the last formula to the spaces $\mathfrak{M}_d^{p,q,\sigma}$ with fixed d and fixed $b = d(1/p - 1/q)$. $\square$

Remarks on compactly supported multipliers. The proofs show that for multipliers which are compactly supported away from the origin the result of Theorem 1.2 can be sharpened.

Theorem 10.2. *Let m be compactly supported and integrable in $(0,\infty)$. Suppose $1<d<\infty$, $1<p<\frac{2d}{d+1}$, $p\le q<2$ and $1\le\sigma\le\infty$. Then the following statements are equivalent.*

(i) T_m maps $L^{p,\sigma}(\mu_d)$ boundedly to $L^{q,\sigma}(\mu_d)$.

(ii) T_m maps $L^{p,1}(\mu_d)$ boundedly to $L^{q,\sigma}(\mu_d)$.

(iii) $\|\mathcal{B}_d[m]\|_{L^{q,\sigma}(\mu_d)}<\infty$.

(iv) $\big\|(1+|\cdot|)^{-\frac{d-1}{2}}\mathcal{F}_{\mathbb{R}}^{-1}[m]\big\|_{L^{q,\sigma}((1+|x|)^{d-1}dx)}<\infty$.

A similar statement can be formulated for the analogue of Theorem 1.1 (again for m supported in $(1/2,2)$). In particular for the case $\sigma=\infty$ we see that then the restricted weak type (p,p) inequality, the weak type (p,p) inequality and the stronger $L^{p,\infty}_{\mathrm{rad}}\to L^{p,\infty}_{\mathrm{rad}}$ bound are all equivalent in the range $1<p<\frac{2d}{d+1}$. We note that for the case of Bochner-Riesz multipliers such endpoint $L^{p,\infty}_{\mathrm{rad}}$ bounds had been obtained by Colzani, Travaglini and Vignati [9], extending earlier weak type endpoint bounds by Chanillo and Muckenhoupt [6]. The result for Bochner-Riesz means follows from the above theorem (after separately dealing with the irrelevant part of the multiplier near 0). This phenomenon has no analogue for Fourier multipliers on $\mathbb{R}^d$ since $L^{p,\infty}\to L^{p,\infty}$ boundedness for translation invariant operators on $\mathbb{R}^d$ already implies $L^p\to L^p$ boundedness ([8], [30]).

The proof of Theorem 10.2 is essentially the same as the proof of Theorem 1.2, but more elementary since only a finite number of dyadic scales on the multiplier side are involved hence no Littlewood-Paley theory and singular integral estimates are needed. The difference (and improvement) in condition (i), and the extended range of σ come from Proposition 5.1 which involves only one dyadic scale and the space $L^{p,\infty}(\mu_d)$ on the right hand side of (5.10).

11. Open problems

11.1. *Radial Fourier multipliers.* Let K be a radial convolution kernel on $\mathbb{R}^d$, $d\ge 2$.

Question: Is there a $p>1$ for which the condition (1.1) (with $\sigma=p$) implies that the convolution operator $f\mapsto K*f$ is bounded on $L^p(\mathbb{R}^d)$?

The local version of this is open as well:

Question: Suppose that K is radial and $\widehat{K}$ is compactly supported in $\mathbb{R}^d \backslash \{0\}$. Is there a $p > 1$ for which the condition $K \in L^p(\mathbb{R}^d)$ implies that the convolution operator $f \mapsto K * f$ is bounded on $L^p(\mathbb{R}^d)$?

It is known (*cf.* [22]) that under a slightly weaker condition than (1.1), namely the finiteness of $\sup_{t>0} \|\Phi * K_t\|_{L^p((1+|x|)^\varepsilon)}$ for some $\varepsilon > 0$ implies L^p boundedness for certain $p > 1$. The condition on p is that for the dual exponent p' the local smoothing problem for the wave equation in $\mathbb{R}^{d+1}$ can be solved up to endpoint estimates. Wolff [37] proved such estimates for $d = 2$ and large p'; for corresponding results in higher dimensions see [20], and for the currently known ranges of Wolff's inequality see [13].

It is likely that in order to prove or come closer to a characterization one needs to prove an endpoint version of Wolff's inequality. The currently known method of proof (by induction on scales) fails to give such sharp bounds.

11.2. *Localized Besov conditions.* Short of a characterization one can ask whether for some $p > 1$ the L^p condition of Corollary 1.5

$$\sup_{t>0} \|\varphi m(t\cdot)\|_{B^2_{d(\frac{1}{p}-\frac{1}{2}),p}} < \infty$$

implies that $m(|\cdot|)$ is a multiplier of $\mathcal{F}L^p(\mathbb{R}^d)$. Again the analogous question for m supported in $(1/2, 2)$ is also open. A result which comes close is in [29]. There a scale of spaces $R^p_{\alpha,s}$ is introduced with $B^p_{\alpha,1} \subset R^p_{\alpha,s} \subset B^p_{\alpha,p}$ for $1 \le s \le p$ and $L^p(\mathbb{R}^d)$ boundedness is proved under the condition $\sup_{t>0} \|\varphi m(t\cdot)\|_{R^2_{d(1/p-1/2),p}} < \infty$, for $1 < p \le \frac{2(d+1)}{d+3}$.

11.3. *Localized multiplier conditions.* Does the analogue of Corollary 1.3 hold for radial Fourier multipliers, acting on general functions in $L^p(\mathbb{R}^d)$, some $p > 1$?

11.4. *Hankel multipliers in the complementary range.* No nontrivial characterization just in terms of the convolution kernel seems to be known (and perhaps may not be expected) for the range $\frac{2d}{d+1} \le p < 2$.

Addendum, January 2008. Very recently, after the submission of this paper, F. Nazarov and the second author made some progress concerning the problems on radial Fourier multipliers. The manuscript [23] contains characterizations for given p, provided that the dimension d is large enough. Presently, there are no optimal results concerning the range of p; moreover, in dimensions $2, 3, 4$, the problems are still open for any $p \in (1, 2)$.

Appendix A. On Zafran's result

Recall that for a compatible couple of Banach spaces (A_0, A_1) a space $X \subset A_0 + A_1$ is called an interpolation space for (A_0, A_1) if there is a constant C so that for every $T : A_0 + A_1 \to A_0 + A_1$ which is bounded on A_0 and bounded on A_1 we have

$$\|T\|_{X\to X} \le C \max\big\{\|T\|_{A_0\to A_0}, \|T\|_{A_1\to A_1}\big\}. \tag{A.1}$$

Zafran [38] showed that the space $M^p(\mathbb{R})$ is not an interpolation space for the pair $M^1(\mathbb{R})$ (the Fourier transforms of bounded Borel measures) and $M^2(\mathbb{R}) = L^\infty(\mathbb{R})$. His arguments in conjunction with Bourgain's theorem on $\Lambda(p)$ sets can be extended to show

Proposition A.1. *Let $1 \le p_0 < p < p_1 \le 2$. Then $M^p(\mathbb{R})$ is not an interpolation space between $M^{p_0}(\mathbb{R})$ and $M^{p_1}(\mathbb{R})$.*

We start quoting a standard result on random Fourier series due to Salem and Zygmund ([26], ch. IV); it can be proved from the distribution inequality for Rademacher expansions and Bernstein's inequality for trigonometric polynomials. Let r_k be the sequence of Rademacher functions and define

$$F_R(t,\theta) = \sum_{k=1}^{R} a_k e^{ik\theta} r_k(t).$$

Then there is a constant C so that for all integers $R \ge 2$, and for $1 \le \rho < \infty$

$$\Big(\int_0^1 \sup_\theta |F_R(t,\theta)|^\rho dt\Big)^{1/\rho} \le C\sqrt{\rho \log R}\big(\sum_k |a_k|^2\big)^{1/2}. \tag{A.2}$$

By the standard averaging argument the $\log R$ term may be dropped if the supremum in θ is replaced by an L^ρ norm.

The proof of Proposition A.1 relies on a deep result by Bourgain [3] (proved earlier by Rudin [25] for p' an even integer).

Bourgain's theorem. *Let $1 < p \le 2$, $p' = p/(p-1)$. There is a constant C_p with the following property. For each integer $N \ge 2$ there exists a set S_N of cardinality N which consists of integers in $[0, N^{p'/2}]$ so that*

$$\Big(\int_0^{2\pi} \Big|\sum_{k\in S_N} a_k e^{ikx}\Big|^{p'} dx\Big)^{1/p'} \le C_p\Big(\sum_k |a_k|^2\Big)^{1/2}. \tag{A.3}$$

In what follows we shall always fix p and the associated family of sets S_N for which (A.3) holds. A consequence of (A.3) is that

$$\Big\|\sum_{k\in S_N} b_k \eta(\cdot - k)\Big\|_{M^p(\mathbb{R})} \le C(p) \sup_k |b_k| \tag{A.4}$$

where η is the Fejér multiplier

$$\eta(\xi) = \begin{cases} 1 - |\xi|, & |\xi| \le 1, \\ 0, & |\xi| > 1. \end{cases}$$

To see (A.4) we first note that (A.3) implies that the sequence $\{b_k \chi_{S_N}(k)\}_{k\in\mathbb{Z}}$ defines a multiplier in $M_2^{p'}(\mathbb{Z})$, and by duality a multiplier in $M_p^2(\mathbb{Z})$, with norms bounded by $C\|b\|_{\ell^\infty}$ (by this we mean that the corresponding convolution operator maps $L^p(\mathbb{T})$ to $L^2(\mathbb{T})$ with norm $\lesssim \|b\|_{\ell^\infty}$). By Hölder's inequality (using the compactness of $\mathbb{T}$) it also follows that this sequence belongs to $M_p(\mathbb{Z})$, since $p \le 2$. Now Jodeit's extension result [19] (see also [12]) for multipliers in $M^p(\mathbb{Z})$ says

$$\Big\| \sum_k m_k \eta(\cdot - k) \Big\|_{M^p(\mathbb{R})} \lesssim \big\| \{m_k\} \big\|_{M^p(\mathbb{Z})}, \quad 1 \le p \le q \le \infty. \tag{A.5}$$

Inequality (A.4) follows.

A third ingredient will be a sequence of multipliers h_N which belong to all $M^q(\mathbb{R})$ classes and satisfy the lower and upper bounds

$$\|h_N\|_{M^q(\mathbb{R})} \approx N^{\frac{1}{q} - \frac{1}{2}}, \quad \text{for } 1 \le q \le 2. \tag{A.6}$$

There are many examples of such families, we choose

$$h_N(\xi) = \chi(\xi) e^{iN|\xi|^2}$$

where χ is a smooth function supported in $(1/2, 2)$ which is equal to one on $[3/4, 5/4]$. To see that (A.6) holds true we examine the kernel $K_N = \mathcal{F}^{-1}[h_N]$. By stationary phase arguments we see that $|K_N(x)| \lesssim N^{-1/2}$ for $N/4 \le x \le 4N$ and $|K_N(x)| \ge cN^{-1/2}$ for $3N/2 \le x \le 5N/2$; moreover by integration by parts $|K_N(x)| \le C_L x^{-L}$ for $x \ge 4N$ and $|K_N(x)| \le C_L N^{-L}$ for $x < N/4$. This shows that $\|K_N\|_{L^q} \ge cN^{1/q-1/2}$ and since h_N has compact support this implies the lower bound in (A.6). The kernel calculation also implies the upper bound for $q = 1$ and interpolation with the trivial L^2 bound yields (A.6).

Proof of Proposition A.1. Let N be a large integer, $R \gg N$ and let S_N be a set in $[0, R]$ so that (A.4) holds (by Bourgain's theorem we may choose $R \approx N^{p'/2}$). Essentially following Zafran we then consider the rank one operators $L_N : M^q \to M^q$ defined by

$$L_N(m) = v_N(m) h_N, \quad \text{where } v_N(m) = \frac{1}{N} \sum_{k\in S_N} \rho_k \int m(\xi) \eta(\xi - k) d\xi.$$

Here we assume that $\rho_k \in \{1, -1\}$ are chosen so that

$$\sup_x \Big| N^{-1} \sum_{k\in S_N} \rho_k e^{ikx} \Big| \le CN^{-1/2} \sqrt{\log R}; \tag{A.7}$$

this can be achieved by (A.2).

We shall show that

$$\|L_N\|_{M^q\to M^q} \le C\min\{N^{1/q-1/2}, N^{-1+1/q}R^{1/q'}\sqrt{\log R}\}, \quad 1\le q\le 2; \tag{A.8}$$

moreover if p is as in (A.4) then

$$\|L_N\|_{M^p\to M^p} \gtrsim N^{1/p-1/2}. \tag{A.9}$$

We first show that the validity of (A.8) and (A.9) implies the assertion of the Proposition. Namely if $M^p(\mathbb{R})$ were an interpolation space of $(M^{p_0}(\mathbb{R}), M^{p_1}(\mathbb{R}))$, with $p_0<p<p_1$, then

$$\|L_N\|_{M^p\to M^p} \le \mathcal{C}\max\{\|L_N\|_{M^{p_0}\to M^{p_0}}, \|L_N\|_{M^{p_1}\to M^{p_1}}\}. \tag{A.10}$$

We use the first bound in (A.8) for $q=p_1$ and the second one for $q=p_0$. Thus by (A.10) and (A.9)

$$N^{1/p-1/2} \lesssim \mathcal{C}\max\{R^{1/p_0'}\sqrt{\log R}N^{-1/p_0'},\, N^{1/p_1-1/2}\}.$$

By Bourgain's theorem we may choose N large and $R\approx N^{p'/2}$. Since $p<p_1$, the last displayed inequality implies $1/p-1/2\le(\frac{p'}{2}-1)/p_0'$ which solving for p is equivalent to $p\le p_0$, a contradiction.

Proof of (A.8). We set $\omega_N := N^{-1}\sum_{k\in S_N}\rho_k\mathcal{F}^{-1}[\eta(\cdot-k)]$. Since the Fejér kernel $\mathcal{F}^{-1}[\eta]$ belongs to $L^1\cap L^\infty$ we observe that $\|\mathcal{F}^{-1}[\eta]\|_{L^r}<\infty$ and hence by (A.7)

$$\|\omega_N\|_{L^r(\mathbb{R})} \lesssim N^{-1/2}\sqrt{\log R}, \quad 1\le r<\infty. \tag{A.11}$$

In view of (A.6) the inequality (A.8) follows from

$$|v_N(m)| \lesssim \min\{1, N^{-1/2}R^{1/q'}\sqrt{\log R}\}\,\|m\|_{M^q}, \quad 1\le q\le 2. \tag{A.12}$$

The first bound in (A.12) is obvious since $|v_N(m)|\le\|m\|_\infty$. The second follows from Plancherel's theorem. To see this let $\zeta_R(\xi)=\zeta_0(\xi/R)$ where ζ_0 is compactly supported with the property that $\zeta_0(\xi)=1$ for $|\xi|\le 2$. Then by (A.11)

$$\begin{aligned}|v_N(m)| &= c\Big|\int\mathcal{F}^{-1}[m\zeta_R](x)\omega_N(x)dx\Big|\\ &\le \|\omega_N\|_{q'}\|\mathcal{F}^{-1}[m\zeta_R]\|_q \le \|\omega_N\|_{q'}\|m\|_{M^q}\|\mathcal{F}^{-1}[\zeta_R]\|_q\end{aligned}$$

and the second bound in (A.12) follows if we observe that $\|\mathcal{F}^{-1}[\zeta_R]\|_q = O(R^{1/q'})$. Thus (A.8) is proved.

Proof of (A.9). Here we use (A.4) (which was a consequence of the crucial $\Lambda(p')$ estimate for the set S_N). We apply L_N to $\widehat{\omega}_N$ and obtain

$$\|L_N\|_{M^p\to M^p} \geq \frac{\|L_N(\widehat{\omega}_N)\|_{M^p}}{\|\widehat{\omega}_N\|_{M^p}} = \frac{\|\widehat{\omega}_N\|_2^2\|h_N\|_{M^p}}{\|\widehat{\omega}_N\|_{M^p}} \gtrsim \|h_N\|_{M^p}$$

where we have used that $\|\widehat{\omega}_N\|_2^2 \approx N^{-1}$ and $N\|\widehat{\omega}_N\|_{M^p} \lesssim 1$, by (A.4). Thus (A.9) follows from (A.6). $\square$

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